Information in Direct and Indirect Quantum Measurements

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Abstract

The information efficiency of direct and indirect (generalized) quantum measurements is analyzed. The longstanding controversial problem, whether indirect measurements can yield more information than direct ones, is solved. It is shown that for any quantum system described by an infinite-dimensional separable Hilbert space, the informations attainable by direct and indirect measurements are exactly equal.

The quantum-mechanical generalization of Shannon's measure of information introduced in [1,2] is based on "classical" J. von Neumann's model of quantum measurements. Such measurements are called "direct," since it is assumed that the measuring instrument (classically described) is applied directly to the system whose state we would like to identify (the system that carries information). However, the concept of quantum measurements can be extended by introducing indirect measurements which involve an additional quantum system coupled to the information carrier.

Consider a quantum system with an ensemble of macrostates $S = \{s_i, p_i\}$, each macrostate s_i being described by a density matrix $\hat{\rho}^{(i)}$ in a Hilbert space H_1. Consider an auxilliary quantum system (called in [3] "ancilla") whose state is independent of the state of the system carrying information and described by a density operator $\hat{\rho}^{(o)}$ in a separable Hilbert space H_2. Then the state of the compound system consisting of the two systems together is described by a tensor-product density operator

$$\hat{\hat{\rho}}^{(i)} = \hat{\rho}^{(i)} \otimes \hat{\rho}^{(o)} \tag{1}$$

in the tensor-product Hilbert space

$$H = H_1 \otimes H_2$$

Denote also

$$\hat{\hat{\rho}} = \hat{\rho} \otimes \hat{\rho}^{(o)} = \sum_i p_i \hat{\rho}^{(i)} \otimes \hat{\rho}^{(o)} \tag{2}$$

A complete indirect (generalized) measurement performed over the compound system and associated with a certain orthogonal basis K in the space H.

Definition 1. Information about the macrostate S of the quantum system for a given state of the ancilla in the outcome of an indirect measurement associated with a basis K in the space $H_1 \otimes H_2$ is the quantity

$$J_K = -\sum_k \sum_n \rho_{kn,kn}(K) \ \ln \rho_{kn,kn}(K) + \\ + \sum_i \sum_k \sum_n \rho^{(i)}{}_{kn,kn}(K) \ \ln \rho^{(i)}{}_{kn,kn}(K) \tag{3}$$

where $\rho_{kn,kn}$ and $\rho^{(i)}{}_{kn,kn}$ are diagonal elements of density matrices $\hat{\hat{\rho}}$ and $\hat{\hat{\rho}}^{(i)}$, respectively, in the basis K.

Definition 2. Information about the macrostate S of a physical system obtainable by indirect measurements is the quantity

$$J = \sup_K \sup_{\hat{\rho}^{(o)}} J_K \tag{4}$$

It has been shown by Naimark [4] that any direct measurement, i.e. any orthogonal resolution of identity in H is equivalent to a resolution of identity (generally speaking, non-orthogonal) in H_1, and, conversely, any non-orthogonal resolution of identity in H_1 can be considered as induced by an orthogonal resolution of identity in a certain tensor-product space H. By a non-orthogonal resolution of identity we mean a set of self-adjoint operators $\{B_k\}$ such that each B_k is nonnegative definite

$$B_K \geq \hat{0}, \quad \text{and} \quad \sum_K \hat{B}_K = \hat{1} \tag{5}$$

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but, in general, $\mathrm{Tr}\, B_K B_{K'} \neq 0$

Indirect quantum measurements were discussed in a number of works (e.g. [3, 5-15]). It was shown that randomized, successive and adaptive quantum measurements are equivalent to some indirect measurements [10], and, thus, indirect measurements are the most general kind of quantum measurements. It was also shown [13] that

$$J \leq I_0 \quad (6)$$

where I_0 is the entropy defect of the quantum system [2].

However, it remained a controversial problem for more than ten years, whether indirect measurements can yield more information than the direct ones. In [12] an example was constructed, where, indeed, J>I. This example, however, uses "state vectors" in a two-dimensional Eucilian space and therefore it seems to be not relevant to quantum systems with states described by operators in a Hilbert (infinitely-dimensional) space. Intuitively, the phenomenon that J can exceed I in a finite-dimensional space is due to the fact that in such cases the cardinality |S| of the set of states exceeds the dimensionality d of the space in which they are embedded. (It has been shown by Davies [11] that the cardinality of the set $\{B_K\}$ of self-adjoint operators used in the optimal indirect measurement obeys an inequality

$$d \leq |\{B_K\}| \leq d^2 \quad (7)$$

but, obviously, this result is not applicable in the infinie-dimensional case.)

Finally, we have proved [15] that the situation is, indeed, different in an infinite-dimensional Hilbert space.

Lemma 1. For any set of density operators $\hat{\rho}^{(i)}$, probabilities p_i, and the density operator of the ancilla $\hat{\rho}^{(o)}$, and for any basis K in the product Hilbert space $H_1 \otimes H_2$ there exists a pure state $\rho_1{}^{(o)} = |\psi_1\rangle\langle\psi_1|$ of the ancilla such that

$$J_K\{\hat{\rho}^{(i)}; p_i; \hat{\rho}_1{}^{(o)}\} \geq J_K\{\hat{\rho}^{(i)}; p_i; \hat{\rho}^{(o)}\} \quad (8)$$

This pure state can be always considered as the first basic vector in the initial basis of the Hilbert space H_2 of the ancilla.

Lemma 1 implies that the variation over $\hat{\rho}^{(o)}$ can be omitted in Definition 1

Lemma 2. For any set $\{\hat{\rho}^{(i)}, p_i\}$ and for any basis K in $H = H_1 \otimes H_2$ there exists a set of density operators $\hat{\sigma}^{(i)}$ and a basis R in H_1 such that

$$I_R\{\hat{\sigma}^{(i)}; p_i\} = J_K\{\hat{\rho}^{(i)}; p_i\} \quad (9)$$

Lemma 3. If all the density operators $\hat{\rho}^{(i)}$ can be represented as mixtures of a finite number of pure states, then for any set of p_i and for any basis K in H there exists a basis R in H_1 such that

$$I_R\{\hat{\rho}^{(i)}; p_i\} = J_K\{\hat{\rho}^{(i)}; p_i\} \quad (10)$$

Theorem. For any set of density operators $\hat{\rho}^{(i)}$ in a separable (infinite-dimensional) Hilbert space, for any set of probabilities p_i such that the entropy of the distribution $H = -\sum_i p_i \ln p_i < \infty$ and for any ancilla with a separable Hilbert of Euclidean space

$$I\{\hat{\rho}^{(i)}; p_i\} = J\{\hat{\rho}^{(i)}; p_i\} \quad (11)$$

The theorem shows that in the case of a quantum system associated with a separable Hilbert space direct measurements are as efficient as more general indirect measurements.

This result resolves the long-standing important problem concerning the information efficiency of direct and indirect quantum measurements. Though indirect measurements (such as in homodyne or heterodyne reception of an electromagnetic signal) may be practically convenient, in principle, we cannot gain by using them, compared to direct measurements, when we deal with a quantum system described by an infinite-dimensional Hilbert space. Note that though some degrees of freedom of a quantum system (such as spin and angular momentum) can be described with a finite-dimensional space, the complete description of even a single quantum particle implies an infinite-dimensional space.

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