Space and Time in Computation, Topology and Discrete Physics

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Abstract

A step can be regarded as an elementary ordering of two objects (or operators). A step is a distinction combined with an action that crosses the boundary of that distinction. The elementary step can be seen as a reference, as a division of space or as a tick of a clock. By looking at the structure of a step, we provide a context that unifies specific aspects of special relativity, Laws of Form, topology, discrete physics and logic design.

1 Introduction

This paper considers the structure of a step in the contexts of space, time and computation. A step can be regarded as a distinction, coupled with the action of crossing from one side of that distinction to the other. By taking such an elementary consideration as our theme, we are able to bring together a very wide range of ideas and techniques under one roof and make connections among them.

The second section is a brief discussion of Step, written in non-technical language. Section 3 takes an ordered pair as the model for a step and introduces the notion of iterant. An iterant is an infinite pattern generated from the ordered pair. Such a pattern can be regarded as a spatial entity or as a temporal entity (oscillation). In fact, an iterant is neither spatial nor temporal, although these are two valid ways of viewing such a form.

Section 4 takes Section 3 and expands it, showing that the mathematical structure of special relativity arises naturally in relation to iterants. This leads, in Section 5, to the algebra of special relativity and a discussion of the square root of minus one as a combination of waveforms composed with a time-shift (delay). Section 6 expands on iterants in the boolean context and the role of the square root of negation. At this point we have arrived at a formal notion of imaginary values in logic. These imaginary values all involve special time-shifts in the temporal context or shifts of space in the spatial context. Section 7 goes more deeply into this matter of imaginary values by looking at the subtley of transitions in examples ranging from the game of GO to asynchronous circuit design, the Fixed Point Theorem of the Church-Curry lambda calculus and questions about the nature of observation in quantum mechanics.

Section 8 shows how matrix algebra develops naturally from iterants and illustrates with an example from particle physics. Section 9 discusses how a time-shift readjusts the classical calculus of finite differences. Section 10 is a brief discussion of the role of topology and knot theory in relation to the theme of a step. Finally, section 11 takes the ideas and themes of the paper and discusses them in the light of the design of ideal compilers. A key example is given via Laws of Form of the descent from a higher level language to a more primitive language.

Section 11 can be read independently of the rest of the paper. We have illustrated throughout the paper how seemingly sophisticated ideas and structures are directly related to primitive ideas and forms of language. The ideal complier for human thought will take complexities and reduce them to intelligible and workable simplicities. I hope that this paper provides ground for discussion of this theme.

2 To Take a Step

In order to take a step there has to be a here and a there. We step across the threshold. We take a step onto the surface of the moon. There appears to be a boundary and it is possible to step across that boundary into a new state.

The structure of step and distinction are intertwined. It is, however, possible to harbor a distinction without taking a step. The beginner at the top of the ski slope knows the distinction of altitude full well but finds the act of stepping out onto the slope quite impossible.

It is also possible to take a step without crossing the boundary of a distinction. Zeno steps halfway to the wall again and again, never reaching it, never crossing it. This step creates no permanent crossing, no mark of distinction.

The concepts Step and Distinction are distinct, yet each may be enjoined in the study of the other.

3 The Ordered Pair and a Non-Numerical Small World

Consider the ordered pair $[a, b]$ as representative of a step. The ordered pair represents a step from a to b.

The a precedes the b. At this stage a and b are letter symbols. They can stand for anything.

$[a,b]$ abbreviates a process. Let [a,b] stand for the elementary repetition *...abababab...* . Then $[a,b]$ becomes a "freeze" of that process, a way of viewing it with a precedence of a before b. The vibration says *...ab..ab..ab..ab....* It could just as well be the other way. $[b,a]$ stands for the same process, heard differently as *...ba..ba..ba..ba..ba....* Two ordered pairs correspond to two possible observations of the same process. Here we have a simple model of a world ...ababababab... and two possible modes of observation of that world: $[a,b]$ and $[b,a]$.

The frieze pattern or *iterant* *...abababab...* is intrinsically a whole infinite form and not inherently spatial or temporal. The interpretation of an iterant as a sequential process is a convenient way of speaking, but the iterant is mathematically prior to the concepts of time and of space.

By itself the iterant is not distinct from itself but it can be juxtaposed with a copy of itself in such a way that it and the copy are seen to be distinct from one another.

...abababababab...

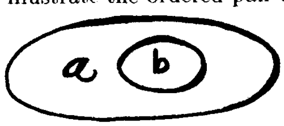
...babababababa...

There are two possible modes of observation of the one iterant. The two modes are complementary. One cannot be seen without excluding the other. This complemenarity arises almost paradoxically from the fact that the iterant itself does not "have" these two states. The states arise through the juxtaposition of the iterant with itself.

$[a,b]$ and $[b,a]$ are descriptions of the process *...ababababab....* In order to have a scientific theory in this small world of one iterant and two descriptions we need a way to extract invariant information from the descriptions. The simplest invariant is non-numerical. It is the unordered pair or set corresponding to the ordered pair. Thus we let $I[a,b] = \{a,b\}$ so that $\{a,b\} = \{b,a\}$ and hence $I[a,b] = I[b,a]$.

The ordered pair $\{a,b\}$ is traditionally defined in terms of sets by the equation $[a,b] = \{a,\{b\}\}$. The set-theoretic construction of order places the two elements to be ordered at different levels in the heirarchy of membership.

In order to get from a to b in $\{a,\{b\}\}$ one must cross the boundary (curly bracket) that separates them. In fact, drawing boundaries instead of the set brackets, we could illustrate the ordered pair as shown below.



Each element (a or b) of the pair becomes a label for one side of a distinction. That distinction is drawn in the plane and its outer compartment is labelled a and surrounded by another boundary marker. The distinction whose sides a and b discriminate is made within the outer circle.

4 Special Relativity

It may happen that a and b are entities (like numbers) that admit a rule of combination (call it multiplication) indicated by their juxtaposition ab so that $ab = ba$ and $(ab)c = a(bc)$. In this case we can construct an invariant $\Delta[a,b]$ by the equation $\Delta[a,b] = ab$. Then $\Delta[a,b] = \Delta[b,a]$. Ordering is released through the commutativity of multiplication.

Let us suppose that there are "numbers" k that admit inverses k^{-1} so that $kk^{-1} = 1$ where $1a = a1 = a$ for any a. Then we have the equation

$$\Delta[ka, k^{-1}b] = \Delta[a,b].$$

Proof. $\Delta[ka,k^{-1}b] = (ka)(k^{-1}b) = ((ka)k^{-1})b = (k(ak^{-1}))b = (k(k^{-1}a))b = ((kk^{-1})a)b = (1a)b = (a)b = ab.$

With algebraic structure of this kind present, we have a group of transformations of the form

$$T_k[a,b] = [ka, k^{-1}b]$$

leaving the "value" $\Delta[a,b]$ invariant. The formal structure of this group of transformations is identical to that of the Lorentz group in the special relativity of one time dimension and one space dimension. The mathematical structure of special relativity arises almost automatically from the consideration of a single iterant!

The relationship to special relativity is as follows: Let

$$a = t - x$$

and

$$b = t + x$$

where t denotes the time coordinate and x the space coordinate. Let the speed of light be taken as $c = 1$ by convention. Then the invariant interval of relativity is

$$\Delta = c^2t^2 - x^2 = t^2 - x^2 = (t-x)(t+x) = ab.$$

Thus $\Delta = \Delta[a,b] = ab$ is the invariant interval for special relativity. T_k is neccessarily a Lorentz transformation written in the "radar" coordinates $a = t-x$, $b = t+x$. (See also [6] and [18], [19].)

The pair $[a,b] = [t-x, t+x]$ consists in the emission and reception times of a signal sent from the observer that intercepts the event and is returned to the observer. Since this is a description of the radar process, these are called radar coordinates.

It is worth pointing out that this same group of transformations arises from simple valuation. To wit: Consider a distinction whose sides are assigned values a and b by one observer and a' and b' , respectively, by another observer. Assume that these values are real numbers and that they are not zero. Then there exist constants R and S such that $a' = Ra$ and $b' = Sb$. These constants express the relative differences in evaluation of the two sides of the distinction for the two observers. Let

$$\rho = RS$$

and

$$k = \sqrt{R/S}.$$

Then $R = \rho k$ and $S = \rho k^{-1}$. We can write

$$[a', b'] = [\rho k a, \rho k^{-1} b] = \rho T_k[a, b].$$

Two evaluations of the sides of a distinction are related, up to a scale factor ρ, by an element T_k of the Lorentz group. In this sense, ***special relativity enters into the structure of almost every human or computational interaction.***

5 Iterant Algebra

Define

$$[a, b][c, d] = [ab, cd]$$

and

$$[a, b] + [c, d] = [a + c, b + d].$$

The operation of juxtapostion is multiplication while + denotes ordinary addition. These operations are natural with respect to the structural juxtaposition of iterants:

...abababababab...

...cdcdcdcdcdcd...

Structures combine at the points where they correspond. Waveforms combine at the times where they correspond. Iterants conmbine in juxtaposition. If @ denotes any form of binary compositon for the ingredients (a,b,...) of iterants, then we can extend @ to the iterants themselves by the definition $[a, b]@[c, d] = [a@b, c@d]$. In this section we shall first apply this idea to Lorentz transformations, and then generalize it to other contexts.

So, to work: We have

$$[t - x, t + x] = [t, t] + [x, -x] = t[1, 1] + x[1, -1].$$

Since $[1, 1][a, b] = [1a, 1b] = [a, b]$ and $[0, 0][a, b] = [0, 0]$, we shall write

$$1 = [1, 1]$$

and

$$0 = [0, 0].$$

Let

$$\sigma = [-1, 1].$$

σ is a significant iterant that we shall refer to as a *polarity*. Note that

$$\sigma\sigma = 1.$$

Note also that

$$[t - x, t + x] = t + x\sigma.$$

Thus the points of spacetime form an algebra analogous to the complex numbers whose elements are of the form $t + x\sigma$ with $\sigma\sigma = 1$ so that

$$(t + x\sigma)(t' + x'\sigma) = tt' + xx' + (tx' + xt')\sigma.$$

In the case of the Lorentz transformation it is easy to see the elements of the form $[k, k^{-1}]$ translate into elements of the form

$$T(v) = [(1+v)/\sqrt{(1 - v^2)}, (1-v)/\sqrt{(1 - v^2)}] = [k, k^{-1}].$$

Further analysis shows that v is the relative velocity of the two reference frames in the physical context. Multiplication now yields the usual form of the Lorentz transform

$$T_k(t + x\sigma) = T(v)(t + x\sigma)$$
$$= (1/\sqrt{(1 - v^2)} - v\sigma/\sqrt{(1 - v^2)})(t + x\sigma)$$
$$= (t - xv)/\sqrt{(1 - v^2)} + (x - vt)\sigma/\sqrt{(1 - v^2)}$$
$$= t' + x'\sigma.$$

The algebra that underlies this iterant presentation of special relativity is a relative of the complex numbers with a special element σ of square one rather than minus one ($i^2 = -1$ in the complex numbers).

The appearance of a square root of minus one unfolds naturally from iterant considerations. Here is one story along these lines (compare with [12]). Define the "shift" operator D on iterants by the equation

$$D[a, b] = [b, a].$$

Sometimes it is convenient to think of D as a "delay" opeator, since it shifts the waveform *...ababab...* by one internal time step. Now define

$$i[a, b] = \sigma D[a, b] = [-1, 1][b, a] = [-b, a].$$

We see at once that

$$ii[a, b] = [-a, -b] = [-1, -1][a, b] = (-1)[a, b].$$

Thus

$$ii = -1.$$

This is the traditional construction of the square root of minus one in terms of operations on ordered pairs.

It goes back to the work of Sir William Rowan Hamilton in the last century. Here we have described $i[a,b]$ in a *new* way as the superposition of the waveforms $\sigma = [-1,1]$ and $D[a,b]$ where $D[a,b]$ is the "delay shift" of the waveform $[a,b]$. This point of view on i appears in [12],[10],[11],[15],[17]. Interesting variants on the algebra of waveforms are given in [30].

All these remarks apply to contexts more general than the arena of real numbers and ordinary algebra. The following sections indicate applications to boolean algebra,logic design, quaternions,matrix algebra and physics.

6 The Boolean Context

In the boolean context the fundamental iterants are

$$I = [1,0]$$

and

$$J = [0,1]$$

corresponding to the underlying waveform

$$...010101010101....$$

Elsewhere [15],[17] we have characterized these iterants as "imaginary boolean values". In that context, the waveform is regarded as an oscillation corresponding to the apparently paradoxical equation

$$P = Neg(P).$$

There are many ways to create a context for this equation. The oscillation comes about by regarding the equals sign as a sign of replacement so that $P = 1$ must be replaced by $P = 1' = 0$, then $P = 0$ must be replaced by $P = 0' = 1$ and so on. The result is an oscillating sequence of values that can start with 1 or 0 depending on the initial conditions. This gives rise to I and J respectively. We wish, however to understand P as a (mathematical) entity prior to time and to space. The doubly infinite sequence ...01010101... has just this property if we decide that

$$Neg(...abababab...) = ...a'b'a'b'a'b'a'b'...$$

then, letting

$$P = ...010101010101...,$$

we have Neg(P) = P.

If we just negate coordinatewise, I and J are no longer invariant under negation. We will have $Neg(I) = J$ and $Neg(J) = I$. However, if we *include a delay* in this operation of negation, defining

$$\overline{Neg}[a,b] = NegD[a,b] = [b',a'],$$

then

$$\overline{Neg}I = I$$

and

$$\overline{Neg}J = J.$$

Including I and J in a context of algebraic logic then leads to a multiple valued logic with four values [10].

The subject does not rest in multiple valued logic. There is a new method to keep imaginary values of this kind directly in a boolean context due to the author and James Flagg [12], [23]. In this ***Flagg Resolution*** we require the ***simultaneous*** substitution of P' for P for all instances of P in a given equation. This resolution of paradox is a direct abstraction of the temporal interpretation of P, but it does not require that interpretation. Thus P can retain its purity beyond time and still participate in the boolean framework - a well-appreciated compromise.

This ends our sketch of the boolean context. The next section elaborates on imaginary values.

7 Imaginary Values in Circuit Transition

If the reader is familiar with the game of Go, she will appreciate the subtlety of the capture of a group of stones (the pieces in Go are called stones). In order to capture a group it must be surrounded by stones of the opposite color. To surround a group is to eliminate all its liberties, where a liberty is a possible placement of a stone that is adjacent to other stones of the group. Ordinarily, White may not move into a place where she is surrounded by black stones. Such a place is called an *eye* for Black. Such a move would be a suicide.

However, if the act of moving a White stone into an eye completes the elimnination of the liberties of a group of Black stones, then this act captures these stones before they can become the captors. The Black stones disappear, and White wins the group. White's single stone is surrounded, but since she surrounds Black on her move, the value of being surrounded accrues to White and not to Black.

Could it have been otherwise? Indeed it could. Without a rule to decide this condition, the placement of the White stone would create an ambiguity. Perhaps this would be decided by the speed of response of the players. Whowever calls capture first would win the shot. Go is not designed this way. We can say that Go has installed an ***imaginary value*** to monitor each eye and decide in favor of the person on the move in the event of the elimination of liberties. As soon as the capture takes place, the imaginary value vanishes along with the eye. Before the insertion of White's stone into the eye, the imaginary value is not activated. This imaginary value has the tiny duration of the interval between the performance of the move and its completion.

We are familiar (as in Go) with a concept of imaginary value that takes effect just in the act of stepping across a boundary, just in the act of a transition. These values occur in the circuit of special observation of the whole by a part of that whole, influencing the transition and avoiding ambiguity or paradox.

A simple example is the device one can purchase in a novelty shop that consists in a black box with a lever on it. Pull the lever and the box vibrates, and extends a hand that reaches out and pulls the lever back to its original position, returning the box to its quiescent state.

An example from logic design is an extra gate monitoring two sides of a memory (the memory consists of two NOR gates feeding into one another) and sending the NOR of these two states back to one side of the memory. If the memory is in the ambiguous state of output 1 on both of its gates, then this extra monitor will influence the transition, eliminating the ambiguity. The imaginary value that resides at the extra gate is evidenced in this transitional determinism. Imaginary values of this sort occur in abundance in asynchronous circuit design. Good examples are given in Chapter 11 of *Laws of Form* by G. Spencer-Brown [31]. See also [13],[14].

The specific subject of imaginary values in asynchronous circuit design cannot be taken up in any detail in this paper. This view of imaginary values is highly subtle in comparison to the first pass we made through the imaginary values related to boolean and arithmetical waveforms. These simple paradoxical elements arise from the unavoidable transitions of the self-inverting circuit shown below.

The full context of self-observing circuits is wide-open for further study.

Can we compare the imaginary values in circuit transition with the condition of conscious awareness? In conscious awareness the apparent world and the observer of that world appear together. Time flows and there are no gaps. Is this believable? A fantastic tale. Lets try another scenario. World and self arise, world and self arise, world and self arise. No world, no self. They wax and wane together. No self present in the ever-present gaps. Self gone when world gone. No discontinuity. The film is dark between the frames. The projection booth does not exist between the frames. Continuity arises from discreteness in the presence of an imaginary value. And anything can happen "in-between".

This last musing is not far from the structure of observation in quantum mechanics, where the observer must strike the chord of an eigenvalue or a projection to a subspace of Hilbert space. In between, more observations can be filled in by the doctrine of completeness of states and the limiting scenario of all possible paths from here to there appears to follow a smooth differential equation (Schrödinger's equation) in the absence of an observer. In particular, the model for observation is the establishment of eigenstate and eigenvalue -

$$H\psi = \lambda\psi.$$

The formal analogue of the eigenvector is a fixed point J of an operator F. When

$$F(J) = J$$

we have that J is an ***eigenform*** of F. The analog is with eigenvalue $\lambda = 1$. We can make the comparison of logical and conscious experience with quantum formalism at the level of eigenforms. Condsider the following construction Let $J = F(F(F(F(F(...)))))$. Then $F(J) = J$. This version uses an infinite regress to construct a fixed point. Second version: Let G be defined by the equation

$$Gx = F(xx).$$

This equation occurs in a realm where elements can act on themselves and each other. Then, substituting G for x, we find

$$GG = F(GG).$$

Thus F has a fixed point that comes into existence at once without infinite regress. (We have just proved the Fixed Point Theorem of the Church-Curry Lambda Calculus [2],[8].

The two modes of reaching a fixed point are related as soon as the equality sign is seen as an act of substitution. Then the definition $Gx = F(xx)$ is seen as the description of the process of duplicating x and tucking the two copies next to one another inside F. When x is G, the process must repeat at the two adjacent $G's$ and the system undergoes recursion in the direction of the fixed point. But ***in time*** it never gets there. The definitional substitution jumps directly to the structure of self-reference and avoids unending temporal process. See [22].

These formalisms speak directly to consciousness, where we do not notice the gap between successive applications of "I" because when "I" am present, awareness is, and when "I" am absent, awareness is not. Continuity of self-description is accomplished by a trick directly analogous to the substitution leading to the fixed point in the form $GG = F(GG)$. If I can refer, then why not to myself? Am I then separate from myself in order to so refer? The appropriate equality creates the fiction of continuity, and still allows the act of thinking about thinking. Compare with Wittegenstein [34] " ***The limits of my language*** mean the limits of my world. ...I am my world.(The microcosm.)... There is no such thing as the subject that thinks or entertains ideas....The subject does not belong to the world:rather,it is a limit of the world." von Foerster [32], "I am the observed relation between myself and observing myself", Fuller [9] "I seem to be a verb.", Spencer-Brown [31] "We now see that the first distinction, the mark and the observer are not only interchangeable, but,in the form, identical."

The imaginary value in Church-Curry formalism resides in that *context for the use of an equals sign.* If the equals sign is an instruction for substitution of $F(xx)$ for Gx, then with $x = G$ there needs be an overseer to stop the recursion after an appropriate depth else the computation go into an infinite loop. This "counter" is an imaginary value in our sense. If the "counter" is set at "1" then we get

$$GG \longmapsto F(GG)$$

and the process stops. Here self-reference occurs in that GG refers to a statement involving GG. Thus, perhaps with a little surprise, we see that the pattern of our own self-reference is modelled by a bit of Lambda Calculus coupled with an imaginary value that terminates the recursion as soon as it starts.

We began this digression into the Lambda calculus with a motivation from quantum mechanics - eigenforms as a generalization of eigenstates. Does the formal apparatus of eigenforms inform the matter of quantum observation? Let's turn this question around. Why is quantum mechanics so successful, with a model of observation that seems to be uncoupled with the theory (no underlying mechanism for the act of observation)? If the world were generated from nothing by acts of self-reference there would surely be a way to generate quantum mechanics from a formalism similar to the Lambda Calculus. Such a possibility is not an impossibility. We may wait for such a theory to emerge before the answers to these questions can be found.

8 Matrix Algebra via Iterants

We all think that we know matrix algebra quite well. But it is a recent invention and has some strange wisdom built into its very bones. Look at a 2x2 matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Compare the matrix with the "two dimensional waveform" shown below

....

$...abababababababab...$

$...cdcdcdcdcdcdcdcd...$

$...abababababababab...$

$...cdcdcdcdcdcdcdcd...$

$...abababababababab...$

....

Each matrix freezes out a way to view the infinite waveform.
In order to keep track of this patterning, lets write

$$[a, d] + [b, c]\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The four matrices that can be "framed" in the two-dimensional wave form are all obtained from the two iterants [a,d] and [b,c] via the "delay shift" operation D[x,y] = [y,x] which we shall denote by an overbar as shown below

$$D[x, y] = \overline{[x, y]} = [y, x].$$

Letting $A = [a, d]$ and $B = [b, c]$, we see that the four matrices seen in the grid are

$$A + B\eta, B + A\eta, \overline{B} + \overline{A}\eta, \overline{A} + \overline{B}\eta.$$

The operator η has the effect of rotating an iterant by ninety degrees in the formal plane. Ordinary matrix multiplication can be written in an incredibly concise form using the following rules:

$$\eta\eta = 1$$

$$\eta Q = \overline{Q}\eta$$

where Q is any two element iterant.
For example, let $\epsilon = [-1, 1]$ so that $\overline{\epsilon} = -\epsilon$ and $\epsilon\epsilon = [1, 1] = 1$. Let

$$i = \epsilon\eta$$

then

$$ii = \epsilon\eta\epsilon\eta = \epsilon\overline{\epsilon}\eta\eta = \epsilon(-\epsilon) = -\epsilon\epsilon = -1.$$

We have reconstructed the square root of minus one in the form of the matrix

$$i = \epsilon\eta = [-1, 1]\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

More generally, we see that

$$(A + B\eta)(C + D\eta) = (AC + B\overline{D}) + (AD + B\overline{C})\eta$$

writing the 2x2 matrix algebra in iterant form as a system of hypercomplex numbers. Note that

$$(A + B\eta)(\overline{A} - B\eta) = A\overline{A} - B\overline{B}$$

(Compare with [16].) The formula on the right corresponds to the determinant of the matrix. Thus we define the *conjugate* of $A + B\eta$ by the formula

$$\overline{A + B\eta} = \overline{A} - B\eta.$$

These patterns capture the quaternions, Cayley numbers and generalize to higher dimensional matrix algebra.

It is worth pointing out the first precursor to the quaternions: This precursor is the system

$$\{\pm 1, \pm\epsilon, \pm\eta, \pm i\}.$$

Here $\epsilon\epsilon = 1 = \eta\eta$ while $i = \epsilon\eta$ so that $ii = -1$. The basic operations in this algebra are those of epsilon and eta. Eta is the "delay shift operator" that reverses the components of the iterant. Epsilon negates one of the components, and leaves the order unchanged. The quaternions arise directly from these two operations once we construct an extra square root of minus one that commutes with them. Call this extra root of minus one $\sqrt{-1}$. Then the quaternions are generated by

$$\{i = \epsilon\eta, j = \sqrt{-1}\epsilon, k = \sqrt{-1}\eta\}$$

with

$$i^2 = j^2 = k^2 = ijk = -1.$$

The "right" way to generate the quaternions is to start at the bottom iterant level with boolean values of 0 and 1 and the operation (EXOR). Build iterants on this, and matrix algebra from these iterants. This gives the square root of negation. Now take pairs of values from this new algebra and build 2x2 matrices again. The coefficients include square roots negation that commute with constructions at the next level and so quaternions appear in the third level of this heirarchy. This construction matches the levels of the combinatorial heirarchy [3], [4], [28] and should be compared with the work of Mike Manthey [26].

This construction of the quaternions is discussed in relation to knot theory and the Dirac string trick in the author's book *Knots and Physics* [20]. See also [27]. The fact is that the quaternions, the rotational stucture of 3-space and the structure of spin angular momentum in elementary quantum mechanics are right there in the algebraic description of the properties of a distinction. Sir William Rowan Hamilton called his quaternions the "Science of Pure Time" sixty years before the discovery of special relativity and before the discovery that those quaternions played a crucial role in the structure of spacetime algebra. We have not finished mining the gold in this vein.

8.1 Electrons, Neutrinos and W-Bosons

Here is a vignette of particle physics expressed in iterant algebra. See [7] for a discussion of the weak interactions of elementary particles. $\nu = [1, 0]$ is the *neutrino*. $\overline{\nu} = [0, 1]$ is the *antineutrino*.

$$\nu\nu = \nu$$

and

$$\nu\overline{\nu} = 0.$$

$$e = \overline{\nu}\eta = W^-$$

represents both the *electron* and the W^- *boson*. These two particles are distinct but we can get by in the reactions below by using the same bit of algebra. Finally, the W^+ boson is represented by

$$W^+ = \nu\eta.$$

Here η is our familiar special rotator of the formal plane with $\eta Q = \overline{Q}\eta$ and $\eta\eta = 1$. Then:

$$W^- e = \overline{\nu}\eta\overline{\nu}\eta = \overline{\nu}\nu = 0.$$

$$W^- \nu = \overline{\nu}\eta\nu = \overline{\nu}^2\eta = \overline{\nu}\eta = e.$$

$$W^+ e = \nu\eta\overline{\nu}\eta = \nu^2 = \nu.$$

$$W^+ \nu = \nu\eta\nu = \nu\overline{\nu}\eta = 0.$$

This is an exact catalog of the allowed and not-allowed (0) interactions of these particles. It is an on-going research project to express the rest of standard-model particle physics in these combinatorial and interactional terms through the use of iterants and the concepts of discrete time and space.

9 A Discrete Ordered Calculus

We now turn to one of the simplest contexts for a step. This is the calculus of discrete differences. Let

$$DX = X' - X$$

define the discrete derivative of a variable X whose succesive values in discrete time are

$$X, X', X'', X''', ...$$

We can proceed to do calculus in this realm. An early exercise reveals the formula

$$D(XY) = X'D(Y) + D(X)Y.$$

Proof.

$$D(XY) = X'Y' - XY = X'Y' - X'Y + X'Y - XY$$

$$= X'(Y' - Y) + (X' - X)Y = X'D(Y) + D(X)Y.$$

The key point is that this formula is ***different*** from the usual formula in Newtonian calculus by the time shift of X to X' in the first term. In [25] and [24] we undertake to correct this discrepancy in the calculus of finite differences by taking the derivative D as an *instruction to shift the time to its left.* That is we take $XD(Y)$ quite literally as *first find* DY, *then find the value of* X. In order to find $D(Y)$ the clock must advance one notch. Therefore X has advanced to X' and we have that the *evaluation* of $XD(Y)$ is

$$X'(Y - Y).$$

In order to keep track of this non-commutative time-shifting, Tom Etter and I [24] write $D(X) = [X' - X]$ where the bracket [ ] is a special time- shifter satisfying the properties

$$A[\ ] = [A'] = [\ ]A'$$

$$[AB] = [A]B$$

$$A[B] = [A'B]$$

$$[A + B] = [A] + [B]$$

The time-shifter acts to automatically evaluate expressions in this non-commutative calculus of finite differences that we call DOC. The key result is the adjusted formula:

$$D(XY) = XD(Y) + D(X)Y.$$

Proof.

$$D(XY) = [X'Y' - XY] = [X'Y' - X'Y + X'Y - XY]$$

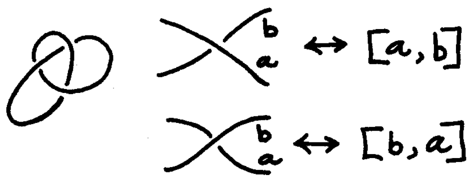
$$= [X'(Y'-Y) + (X'-X)Y] = [X'(Y'-Y)] + [(X'-X)Y]$$

$$= X[(Y'-Y)] + [(X'-X)]Y = XD(Y) + YD(X).$$

The upshot is that DOC behaves formally like infinitesimal calculus and can be used as a foundation for discrete physics. In [25] Pierre Noyes and the author use this foundation to build a derivation of electromagnctism from the formalism of quantum mechanics. The Kauffman Etter calculus is suitable for symbolic computation and can even be used to keep track of the myriad time shifts in the classical calculus of finite differences. This is a case of problems in one field leading to payoffs in another.

10 Topology

There are many contacts between the material in this essay and topological studies. We shall mention one. A knot is encoded in a planar diagram with *crossings* as indicated below. These crossings are extra



structure on a four-valent vertex in the plane and can be encoded by the symbols [A,B] and [B,A] as shown above. Thus a knot or link is a network of iterant pairs. This generalization of flat crossings to pairs leads to a significant generalization of the Penrose spin networks [29] to a formalism where these generalized networks encode three dimensional manifolds and concurrently compute topological invariants [20], [21]. The spin networks are closely related to the combinatorial structure of the representations of the quaternion group SU(2). Many questions are open about the relationships of these structures with quantum field theory [33] and quantum gravity [1]. This section is only a hint, but topology figures in the background of all the structures that we have discussed in this essay.

11 On Constructing a Compiler

A compiler takes a text written in a "higher order language" and translates it into a text written in a language that is more elementary. More elementary usually means fewer primitives and that certain irreducible constructions in the higher order language are constructed from these primitives. Along with the problems of designing compilers there is the problem of discovering perspicuous "lower level" languages. A compiler may not actually translate into machine language, but simply into some other language and the notion of machine language may eventually evolve.

Sometimes, we find remarkable languages that demand attention just because a "higher order" language can be translated into them, and they seem to have fundamental properties of their own. In this section, I want to illustrate how one can "fall into" a primitive language by letting go of certain features in a higher level of description. Please consider the following story:

Let us suppose that we have a distinction. The two sides of this distiction are called "Inside" (I) and "Outside" (O). We allow that it is reasonable to adopt the convention that "The value of a name called again is the value of the name.", or in symbols

$$II = I$$

and

$$OO = O.$$

We also devise a symbol, $<>$, to denote crossing from one side of the distinction to the other side. In this symbolism, when A denotes the value of the given side, then the value of the side obtained upon crossing from A is denoted $< A >$ (read " A cross"). Thus we have the equations

$$< I > = O$$

and

$$< O > = I.$$

Now it occurs to us that we might not actually need to use a symbol for the inside. Why not leave the inside unmarked? In the algebraic language, an empty word, a blank space will indicate the inside. We introduce the *unmarked state* and refer to the outside (O) as the *marked state*. Will the equations still work? We write them down:

$$=$$

$$OO = O$$

$$<\quad> = O$$

$$< O > = \quad .$$

The most interesting of these new equations are the last two. The first, $<> = O$, reads "A crossing from the unmarked state yields the marked state.", and the second reads "A crossing from the marked state yields the unmarked state.". The first equation suggests that we can eleminate the name (O) of the marked state and replace it by the symbol "cross" - $<>$. That is we can take as the *name* of the outside the *instruction* "cross from the unmarked state". This reduces us to two equations:

$$<><> = <>$$

$$<<>> = \quad .$$

The first reads "The value of a call made again is value of the call.". The second reads "The value of a crossing from the marked state is unmarked.". There is really a third equation:

$$<\quad> = <>,$$

reading "The value of a crossing made from the unmarked state is the marked state.". Here we have used the two interpretations of <> for the two sides of the equality sign.

The reader familiar with Spencer-Brown's *Laws of Form* [31] will recognise that our descent from a named distinction has lead us directly and inevitably to the Calculus of Indications. We did not start out in boolean algebra. Neither of the values O or I dominated the other. But in the primitive system of the mark, the marked state naturally dominates the unmarked state and boolean patterns prevail. As a result, the Calculus of Indications and its algebra, The Primary Agebra, become primitive languages into which much of standard logic can be rewritten. I should like to say "compiled".

In the case of this Spencer-Brown compiler for logic and distinctions, the advantages are a little different from the usual situation with machine language. The "primitive" language of the Calculus of Indications is in fact also highly sophisticated. It speaks in the argot of fundamental distinction and uses the power of the unmarked state. The user is well advised to learn to speak this primitive language herself, and to examine the consequences of ascending and descending from the realm of fundamental distinction to more highly articulated speech. In the case of the Calculus of Indications the analogy with compilers should be augmented with the analogy of elementary particles. The marked and unmarked states are the elementary particles of logic, thought,perception. Reduction to expressions in patterns of markedness and void is not void of meaning. The operation of the compiler is to expand world and possibility.

The point I am trying to make and hoping to explore through these examples and through the other structures in this paper is that *the ideal compiler is a* two way *translater of languages*. That compiler should enable us to move freely back and forth among different levels of structure. So far, it has been extraordinarily convenient to have compilers that let us write in high level computer languages. We all appreciate the virtues and flexibilities of simplicity. Most computer users are to a great extent locked away by these very conveniences from lower levels which hold the possibility of being higher levels. Boole entitled his work "The Laws of Thought" [5]. He meant it. He meant to explore the deepest possible elements of human thought. We are still in that quest. It is a quest for a fully open two-way compiler for conscious and unconscious language and thought. It is , I believe, *identical* to the quest of the physicist, natural scientist, mathematician or philosopher who gropes for the keys to fundamental theory and secrets of universe.

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