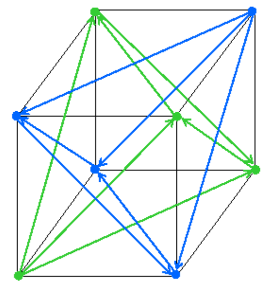

Quantum Geometric Algebra

ANPA Conference

Cambridge, UK

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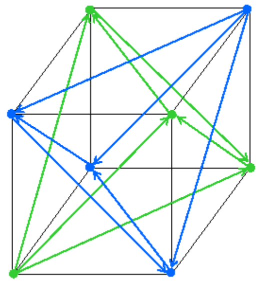


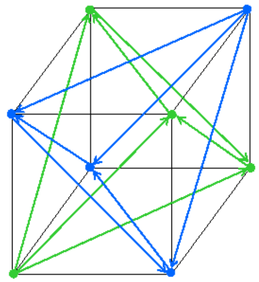
Abstract

Quantum computing concepts are described using geometric algebra, without using complex numbers or matrices. This novel approach enables the expression of the principle ideas of quantum computation without requiring an advanced degree in mathematics.

Using a topologically derived algebraic notation that relies only on addition and the anticommutative geometric product, this talk describes the following quantum computing concepts:

> bits, vectors, states, orthogonality, qubits, classical states, superposition states, spinor, reversibility, unitary operator, singular, entanglement, ebits, separability, information erasure, destructive interference and measurement.

These quantum concepts can be described simply in geometric algebra, thereby facilitating the understanding of quantum computing concepts by non-physicists and non-mathematicians.



Overview of Presentation

- Co-Occurrence and Co-Exclusion
- Geometric Algebra $\boldsymbol{G}_n$ Essentials
  - Symmetric values, scalar addition and multiplication
  - Graded N-vectors, scalar, bivectors, spinors
  - Inner product, outer product, and anticommutative geometric product
- Qubit Definition is Co-Occurrence
  - Standard and Superposition States, Hadamard Operator, Not Operator
  - Reversibility, Unitary Operators, Pauli Operators, Circular basis
  - Irreversibility, Singular Operators, Sparse Invariants and Measurement
  - Eigenvectors, Projection Operators, trine states
- Quantum Registers
  - Geometric product equivalent to tensor product, entanglement, separability
  - Ebits and Bell/magic States/operators, non-separable and information erasure
  - C-not, C-spin, Toffoli Operators
- Conclusions

Co-Occurrence and Co-Exclusion

a = **+a** = **ON** and
$\bar{\mathbf{a}}$ = **–a** = not **ON**
where **a** + $\bar{\mathbf{a}}$ = 0

$a + \bar{b} = \bar{b} + a$

c - d ⟷ d - c

c - d | d - c

c - d + d - c = 0

(0 means cannot occur)

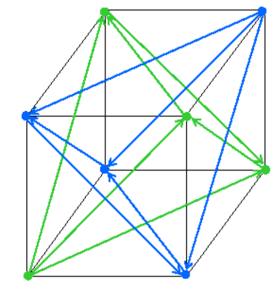
Abstract Space

Co-occurrence means states exist exactly simultaneously

Co-exclusion means a change occurred due to an operator

Abstract Time

Both of Mike Manthey’s concepts used heavily in this research



Boolean Logic using +/* in $\boldsymbol{G}_n$

+	0	1	–1
0	0	1	–1
1	1	–1	0
–1	–1	0	1

*	0	1	–1
0	0	0	0
1	0	1	–1
–1	0	–1	1

→

+	+	–
+	–	0
–	0	+
If same then invert If diff then cancel		

*	+	–
+	+	–
–	–	+
If same then +1 If diff then -1		

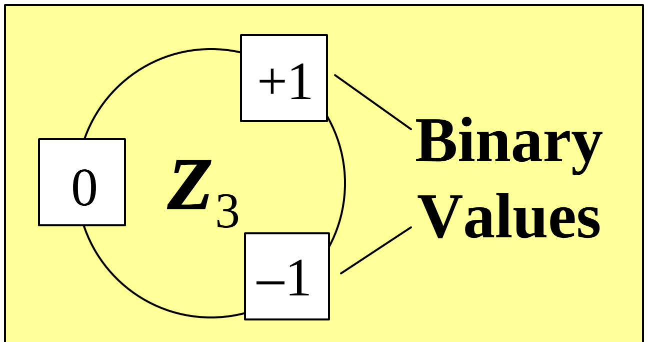
Normal multiplication and mod 3 addition for ring {–1,0,1}, so can simplify to {–,0,+} and remove rows/columns for header value 0.

+ NAND + => –

– NOR – => +

same XNOR *same* => +

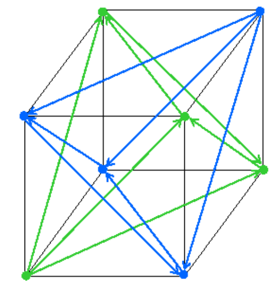
differ XNOR *differ* => –



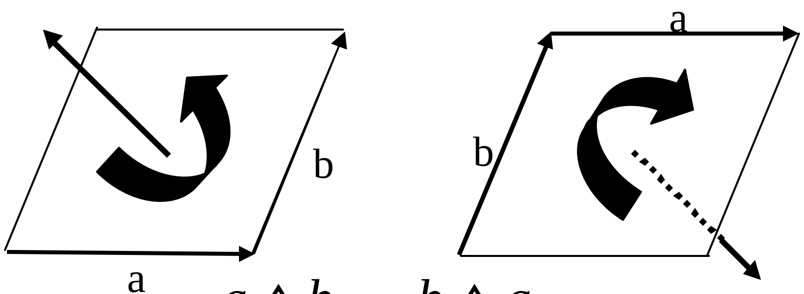
Logic in $\boldsymbol{G}_2$ = span{**a**, **b**}	**GA Mapping** {+, –}	**GA Mapping** {+, 0}
Identity **a**	**a** * 1 = **a** + 0 = **a**	–1 – **a** = – (1 + **a**)
NOT **a**	**a** * –1 = – **a**	–1 + **a** = – (1 – **a**)
a XOR **b**	– **a b**	–1 + **a b**
a OR **b**	**a** + **b** – **a b**	–1 – **a** – **b** + **a b**
a AND **b**	+1 – **a** – **b** – **a b**	+1 + **a** + **b** + **a b**

Also for any vector **e**: since $\mathbf{e}^2=1$ then $\mathbf{e} = 1/\mathbf{e}$

Geometric Algebra is Boolean Complete



Geometric Algebra Essentials



$a \wedge b = -b \wedge a$

$\mathbf{a}\,\mathbf{b} = \mathbf{a}\bullet\mathbf{b} + \mathbf{a} \wedge \mathbf{b}$ where geometric product is sum
$\mathbf{a}\bullet\mathbf{b} = \cos q$ of inner product (is a scalar)
$\mathbf{a} \wedge \mathbf{b} = i \sin q$ and outer product (is a *bivector*)

$\boldsymbol{G}_{n=2}$ generates $N=2^n$: span{**a**, **b**}

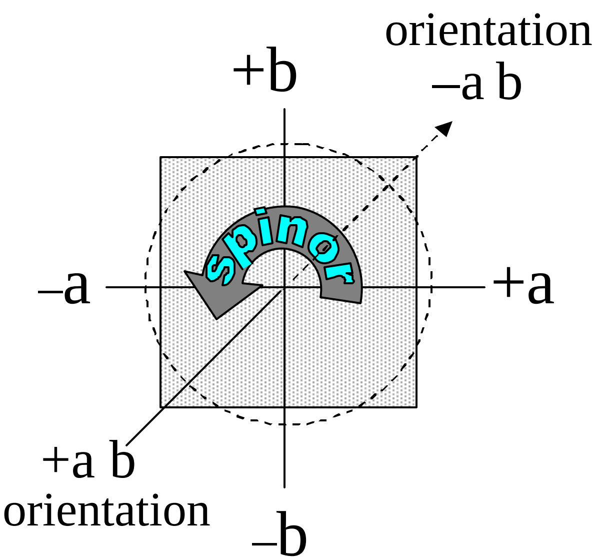
G_2 = scalars {±1}, vectors {a, b}, and bivector {a b} then:

With **a•b** = 0 (only orthonormal basis so are perpendicular)
then **a b** = – **b a** (due to *anti-commutative* outer product)
$\mathbf{a}^2 = \mathbf{b}^2 = 1$ (due to inner product since collinear)

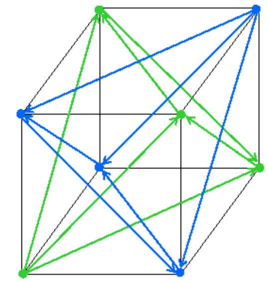
bivector is *spinor* because: (right multiplication by spinor)
a (**a b**) = **a a b** = **b**, and **b** (**a b**) = – **a b b** = – **a**

spinor is also *pseudoscalar I* because:
$(\mathbf{a}\,\mathbf{b})^2 = \mathbf{a}\,\mathbf{b}\,\mathbf{a}\,\mathbf{b} = -\,\mathbf{a}\,\mathbf{a}\,\mathbf{b}\,\mathbf{b} = -(\mathbf{a})^2\,(\mathbf{b})^2 = -1$ = NOT

so $\mathbf{a}\,\mathbf{b} = \sqrt{-1} = \sqrt{NOT}$



also $\mathbf{x}' = R\mathbf{x}\tilde{R}$ with $R = a - b\,\mathbf{a}\,\mathbf{b}, \tilde{R} = a + b\,\mathbf{a}\,\mathbf{b}, a = \cos(q/2), b = \sin(q/2)$



Number of Elements in $\boldsymbol{G}_{\mathrm{n}}$

Graded: scalar, vector, bivector, trivector, …, n-vector for $\boldsymbol{G}_{\mathrm{n}}$ with $N=2^n$ elements

$(1+\mathbf{a})(1+\mathbf{b})(1+\mathbf{c}) = 1 + \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{a\,b} + \mathbf{a\,c} + \mathbf{b\,c} + \mathbf{a\,b\,c}$ <*multivector*

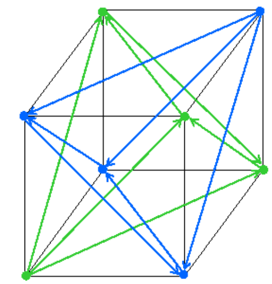
$\boldsymbol{G}_{\mathrm{n}} = \boldsymbol{G}_{\mathrm{n}}^{+} + \boldsymbol{G}_{\mathrm{n}}^{-} = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \langle A\rangle_3 + \ldots + \langle A\rangle_n$

Odd grade terms $\boldsymbol{G}_{\mathrm{n}}^{-} = \langle A\rangle_1 + \langle A\rangle_3 + \ldots$

Even Subalgebra $\boldsymbol{G}_{\mathrm{n}}^{+} = \langle A\rangle_0 + \langle A\rangle_2 + \ldots$

$\boldsymbol{G}_3^{+}$ are the quaternions: $1 + \mathbf{a\,b} + \mathbf{a\,c} + \mathbf{b\,c}$

$Row = n$	$Col = m$	$1+\sum_{m=1}^{n}\binom{n}{m} = N = 2^n$
0	1	$=1$
1	1 **1**	$=2$
2	1 **2** 1	$=4$
3	1 **3** 3 1	$=8$
4	1 **4** 6 4 1	$=16$
5	1 **5** 10 10 5 1	$=32$
6	1 **6** 15 20 15 6 1	$=64$
	Pascal's Triangle	(Binomial)



Inner Product Calculation

$\mathbf{Y} = (\mathbf{x} \wedge \mathbf{y})$ and $\mathbf{Z} = (\mathbf{Y} \wedge \mathbf{z})$ with vector variables **w**, **x**=**a**, **y**=**b**, **z**=**c**

$G_2 = \text{span}\{\mathbf{a},\mathbf{b}\}$: $\mathbf{w} \bullet \mathbf{Y} = \mathbf{w} \bullet (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{w} \bullet \mathbf{a}) \wedge \mathbf{b} - (\mathbf{w} \bullet \mathbf{b}) \wedge \mathbf{a}$

$G_3 = \text{span}\{\mathbf{a},\mathbf{b},\mathbf{c}\}$: $\mathbf{w} \bullet \mathbf{Z} = (\mathbf{w} \bullet \mathbf{a}) \wedge \mathbf{b} \wedge \mathbf{c} - (\mathbf{w} \bullet \mathbf{b}) \wedge \mathbf{a} \wedge \mathbf{c} + (\mathbf{w} \bullet \mathbf{c}) \wedge \mathbf{a} \wedge \mathbf{b}$

Only one non-zero term in sum for *orthogonal* basis set {**a**,**b**,**c**}

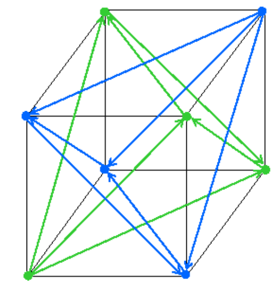
Outer Product

$X \wedge Y$		*Y*			
		+1	**a**	**b**	**a b**
X	+1	+1	**a**	**b**	**a b**
	a	**a**	0	**a b**	0
	b	**b**	–**a b**	0	0
	a b	**a b**	0	0	0

Inner Product

$X \bullet Y$		*Y*			
		+1	**a**	**b**	**a b**
X	+1	0	0	0	0
	a	0	+1	0	**b**
	b	0	0	+1	–**a**
	a b	0	**b**	–**a**	–1

$XY = X \bullet Y + X \wedge Y$ only if *X* or *Y* are assigned vector **x** or **y**



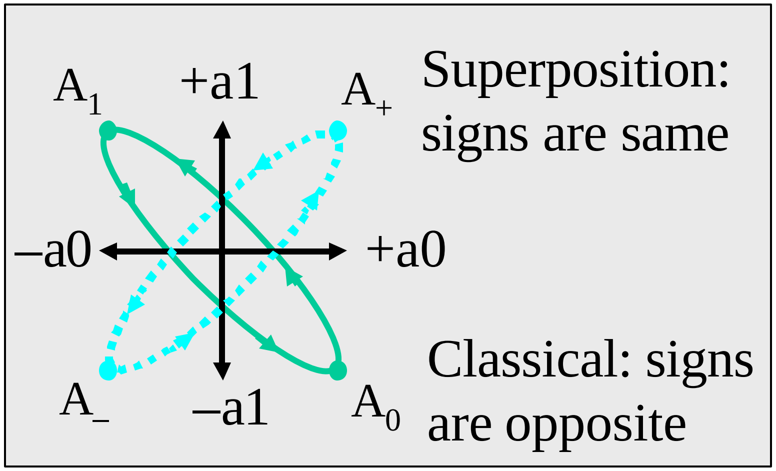
Qubit is Co-occurrence in $\boldsymbol{G}_2$

Single Qubit:

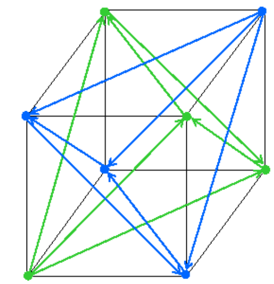
$A = (\pm\mathbf{a0} \pm\mathbf{a1})$

where $\boldsymbol{Q}_1 = \boldsymbol{G}_2 = \text{span}\{\mathbf{a0}, \mathbf{a1}\}$

4 elements & $3^4 = 81$ multivectors



$\boldsymbol{Row}_k$	a0	a1	$A_1 = \overline{\mathbf{a0}} + \mathbf{a1}$	$A_0 = \mathbf{a0} + \overline{\mathbf{a1}}$	$A_+ = \mathbf{a0} + \mathbf{a1}$	$A_- = \overline{\mathbf{a0}} + \overline{\mathbf{a1}}$
R_0	–	–	0	0	+	–
R_1	–	+	+	–	0	0
R_2	+	–	–	+	0	0
R_3	+	+	0	0	–	+
Binary combinations of input states			Anti-symmetric sums are classical states		Symmetric sums are superposition states	
			$A_1 = R_1 - R_2$	$A_0 = R_2 - R_1$	$A_+ = R_0 - R_3$	$A_- = R_3 - R_0$



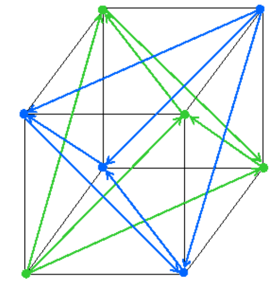
Spinor is Hadamard Operator

Start Phase	Qubit State A	Each Times Spinor	Result = $A\ \mathbf{S}_A$	End Phase
Classical	A_0 = +**a0** – **a1**	+**a0** (**a0 a1**) = +**a1**	A_+ = +**a0** + **a1**	*Superposed*
	A_1 = –**a0** + **a1**	–**a0** (**a0 a1**) = –**a1**	A_- = –**a0** – **a1**	
Superposed	A_+ = +**a0** + **a1**	+**a1** (**a0 a1**) = –**a0**	A_1 = –**a0** + **a1**	**Classical**
	A_- = –**a0** – **a1**	–**a1** (**a0 a1**) = +**a0**	A_0 = +**a0** – **a1**	

Hadamard is the 90° phase or spinor operator $\mathbf{S}_A$= (**a0 a1**)

NOT operator is 180° gate $\mathbf{S}_A^2$ = (**a0 a1**)(**a0 a1**) = –**a0 a0 a1 a1** = –1

Therefore $\mathbf{S}_A = \sqrt{-1} = \sqrt{NOT}$ and generally $\sqrt[r]{q} = q/r$ and $q^p = pq$



Unitary Pauli Noise States in $\boldsymbol{G}_2$

Flip:

Bit

$$s_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Case	Hilbert notation	Use case	GA equivalent is (–1)= complement
[a]	$s_1\|0\rangle \to \|1\rangle$	[a]	(+ **a0** – **a1**)(–1) ➔ (– **a0** + **a1**)
[b]	$s_1\|1\rangle \to \|0\rangle$	[b]	(– **a0** + **a1**)(–1) ➔ (+ **a0** – **a1**)

+

Phase

$$s_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Case	Hilbert notation	Use cases	GA equivalent is spinor $\mathbf{S}_A$ = **a0 a1**
[a]	$s_3\|1\rangle \to -\|1\rangle$	[a]&[b]	(– **a0** + **a1**)(– **a0 a1**) ➔ (+ **a0** – **a1**)
[b]	$s_3\|0\rangle \to \|0\rangle$		
[c]	$-s_3\|1\rangle \to \|1\rangle$	[b]&[c]	(+ **a0** – **a1**)(**a0 a1**) ➔ (+ **a0** + **a1**)

=

Both

$$s_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

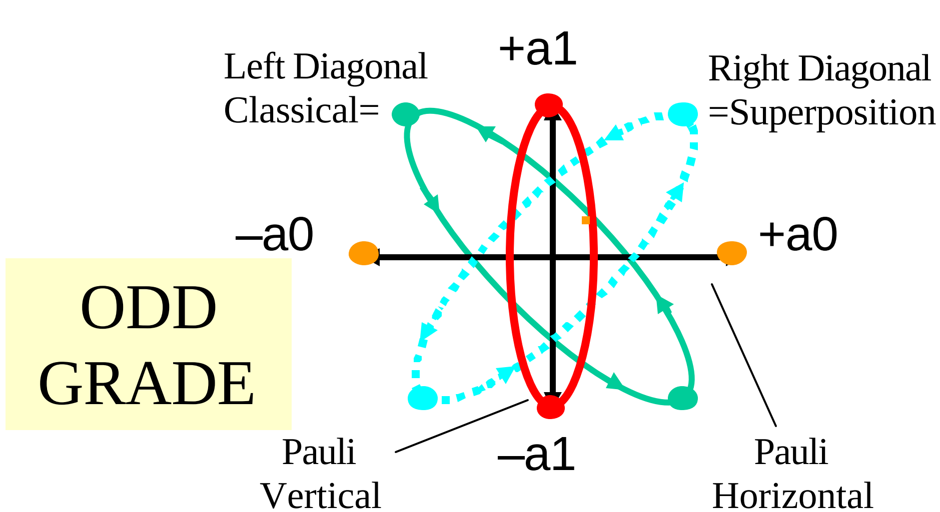
Case	Hilbert notation	Use cases	GA equivalent is (–1 + $\mathbf{S}_A$) = P_A
[a]	$s_2\|0\rangle \to +i\|0\rangle$	[a]&[b]	(+ **a0** – **a1**)(–1 + **a0 a1**) ➔ –**a1**
[b]	$s_2\|1\rangle \to -i\|1\rangle$		

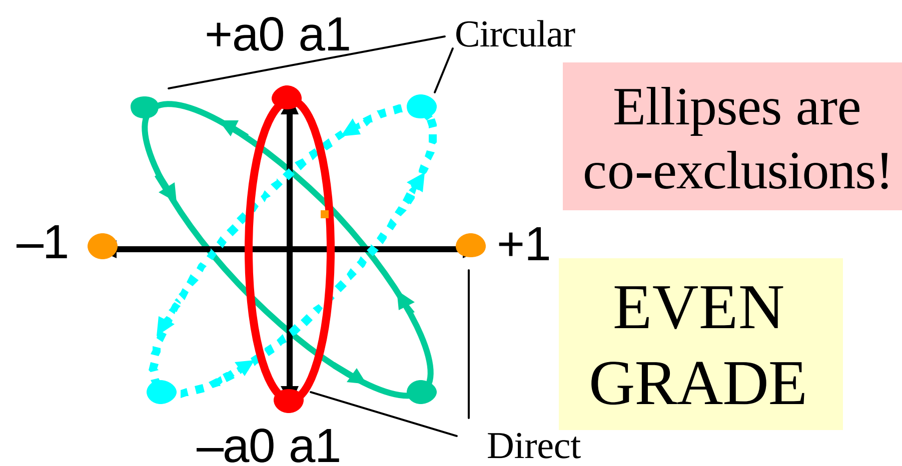
Pauli operators -1, $\mathbf{S}_A$ and P_A are *even* grade!

Reversible Basis Encodings:

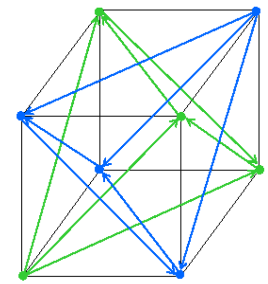
Standard, Dual, Pauli and Circular basis

Label for Row	Start State	Diag (–1 + **a0 a1**)	Diag (**a0**)	Diag (+ **a0** – **a1**)
classical 0	+ **a0** – **a1**	– **a1**	(+1 + **a0 a1**)	–1
classical 1	– **a0** + **a1**	+ **a1**	(–1 – **a0 a1**)	+1
superposition +	+ **a0** + **a1**	+ **a0**	(+1 – **a0 a1**)	+ **a0 a1** *(random)*
superposition –	– **a0** – **a1**	– **a0**	(–1 + **a0 a1**)	– **a0 a1** *(random)*
Label for Basis	Diagonals	Pauli = Ver/Hor	Circular	Direct or Complex
Reversible op. return to start		V/Hor (1 + **a0 a1**)	Cir (**a0**)	Dir (– **a0** + **a1**)





Unitary Operators and Reversibility



For multivector state X and multivector operator Y,

If new state $Z = X\,Y$ then

Y is *unitary* if-and-only-if $W = 1/Y = Y^{-1}$ exists

such that $Y\,W = Y\,Y^{-1} = 1$

Therefore unitary operator Y is *invertible/reversible*:

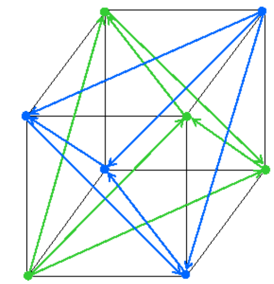
$Z\,/\,Y = X\,Y\,/\,Y = X$

Reversible

For *unitary* Y then *requires* $\det(Y)=\pm 1$ or $|\det(Y)| = 1$

$A_0\,A_1 = 1$
$A_-\,A_+ = 1$

Trines are unitary: $(Tr)^3 = 1$ so $1/Tr = (Tr)^2$
for $Tr = (+1 \pm \mathbf{a0} \pm \mathbf{S}_A)$ or $(+1 \pm \mathbf{a1} \pm \mathbf{S}_A)$



Singular Operators in $\boldsymbol{G}_n$

If $1/X$ is *undefined* then requires $\det(X) = 0$,

Since $(\pm 1 \pm \mathbf{x})^{-1}$ is undefined then $\det(\pm 1 \pm \mathbf{x}) = 0$

and therefore $X = (\pm 1 \pm \mathbf{x})$ is *singular*

Singular examples: $\det(\pm 1 \pm \mathbf{a}) = \det(\pm 1 \pm \mathbf{b}) = 0$

Singular

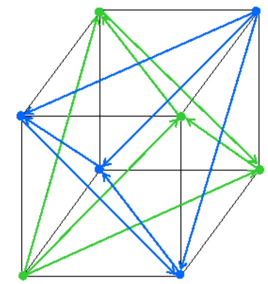
Also fact that: $\det(X)\det(Y) = \det(XY)$,

which means if factor X has $\det(X) = 0$,

then product (XY) also has $\det(XY) = 0$.

In $\boldsymbol{G}_2$: $\det(1 \pm \mathbf{a})\det(1 \pm \mathbf{b}) = \det(1 \pm \mathbf{a} \pm \mathbf{b} \pm \mathbf{a}\,\mathbf{b}) = 0$

$$X^{-1} = (X^*)^T \approx \frac{1}{\det(X)}$$



Row Decode Operators R_k are Singular

Row $_k$	**a0**	**a1**	(–1)(1 – **a0**)	(–1)(1 + **a0**)	(–1)(1 – **a1**)	(–1)(1 + **a1**)
R_0	–	–	+	0	+	0
R_1	–	+	+	0	0	+
R_2	+	–	0	+	+	0
R_3	+	+	0	+	0	+
Summation of R_k ➔			$A0_- = R_0 + R_1$	$A0_+ = R_2 + R_3$	$A1_- = R_0 + R_2$	$A1_+ = R_1 + R_3$
Denoted as Vector ➔			[+ + 0 0]	[0 0 + +]	[+ 0 + 0]	[0 + 0 +]

← Standard Algebraic Notation

Row $_k$	**a0**	**a1**	(1–**a0**)(1–**a1**)	(1–**a0**)(1+**a1**)	(1+**a0**)(1–**a1**)	(1+**a0**)(1+**a1**)
R_0	–	–	+	0	0	0
R_1	–	+	0	+	0	0
R_2	+	–	0	0	+	0
R_3	+	+	0	0	0	+
State logic ➔			$R_0 = A0_- A1_-$	$R_1 = A0_- A1_+$	$R_2 = A0_+ A1_-$	$R_3 = A0_+ A1_+$
Denoted as Vector ➔			R_0 = [+ 0 0 0]	R_1 = [0 + 0 0]	R_2 = [0 0 + 0]	R_3 = [0 0 0 +]

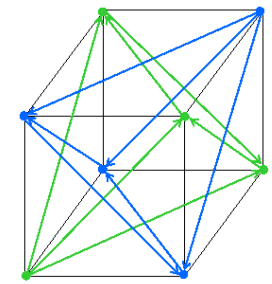
← Dual Vector Notation:

matrix diagonal

$R_0 + R_1 + R_2 + R_3$ = [+ + + +] = 1

R_k are topologically smallest elements in $\boldsymbol{G}_2$ and are linearly independent

Measurement and Sparse Invariants

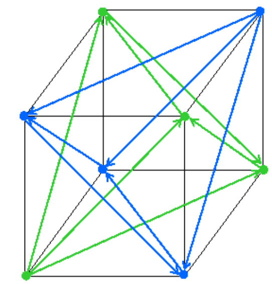


Start States A	Each start state A times each R_k gives the answer			
	$A(1+\mathbf{a0})(1-\mathbf{a1})$	$A(1-\mathbf{a0})(1+\mathbf{a1})$	$A(1+\mathbf{a0})(1+\mathbf{a1})$	$A(1-\mathbf{a0})(1-\mathbf{a1})$
$A_0 = +\,\mathbf{a0} - \mathbf{a1}$	$-1 + \mathrm{a1} = \boldsymbol{I}^{+}$	$+1 + \mathrm{a1} = \boldsymbol{I}^{-}$	$-\mathbf{a0}\,(+1 + \mathbf{a1})$	$+\mathbf{a0}\,(-1 + \mathbf{a1})$
$A_1 = -\,\mathbf{a0} + \mathbf{a1}$	$+1 - \mathrm{a1} = \boldsymbol{I}^{-}$	$-1 - \mathrm{a1} = \boldsymbol{I}^{+}$	$-\mathbf{a0}\,(-1 - \mathbf{a1})$	$+\mathbf{a0}\,(+1 - \mathbf{a1})$
$A_- = -\,\mathbf{a0} - \mathbf{a1}$	$-\mathbf{a0}\,(-1 + \mathbf{a1})$	$+\mathbf{a0}\,(+1 + \mathbf{a1})$	$+1 + \mathrm{a1} = \boldsymbol{I}^{-}$	$-1 + \mathrm{a1} = \boldsymbol{I}^{+}$
$A_+ = +\,\mathbf{a0} + \mathbf{a1}$	$-\mathbf{a0}\,(+1 - \mathbf{a1})$	$+\mathbf{a0}\,(-1 - \mathbf{a1})$	$-1 - \mathrm{a1} = \boldsymbol{I}^{+}$	$+1 - \mathrm{a1} = \boldsymbol{I}^{-}$
End State ➔	$A' \Rightarrow +\,\mathbf{a0} - \mathbf{a1}$	$A' \Rightarrow -\,\mathbf{a0} + \mathbf{a1}$	$A' \Rightarrow +\,\mathbf{a0} + \mathbf{a1}$	$A' \Rightarrow -\,\mathbf{a0} - \mathbf{a1}$
Description ➔	Classical States Measurement		Superposition States Measurement	

$\boldsymbol{I}^{+} \sim +1 \qquad \boldsymbol{I}^{-} \sim -1 \qquad \boldsymbol{I}^{-} = -\boldsymbol{I}^{+} \qquad (\boldsymbol{I}^{\pm})^2 = \boldsymbol{I}^{+}$

$-1 + \mathrm{a1} = [+\,0 + 0] = \boldsymbol{I}^{+0}$ $\qquad +1 - \mathrm{a1} = [-\,0 - 0] = \boldsymbol{I}^{-0}$

$-1 - \mathrm{a1} = [0 + 0\,+] = \boldsymbol{I}^{+90}$ $\qquad +1 + \mathrm{a1} = [0 - 0\,-] = \boldsymbol{I}^{-90}$



Projection Operators P_k and Eigenvectors E_k

Primary Tetrahedron (k=0–3)				Dual Tetrahedron (=7–k)			
k =	$E_k = R_k - 1$	$P_k = -R_k$	$R_k = 1+E_k$	k =	$E_k = R_k - 1$	$P_k = -R_k$	$R_k = 1+E_k$
0	[0 – – –]	[– 0 0 0]	[+ 0 0 0]	7	[0 + + +]	[– + + +]	[+ – – –]
1	[– 0 – –]	[0 – 0 0]	[0 + 0 0]	6	[+ 0 + +]	[+ – + +]	[– + – –]
2	[– – 0 –]	[0 0 – 0]	[0 0 + 0]	5	[+ + 0 +]	[+ + – +]	[– – + –]
3	[– – – 0]	[0 0 0 –]	[0 0 0 +]	4	[+ + + 0]	[+ + + –]	[– – – +]
sum	[0 0 0 0]	[– – – –]	[+ + + +]	sum	[0 0 0 0]	[– – – –]	[+ + + +]

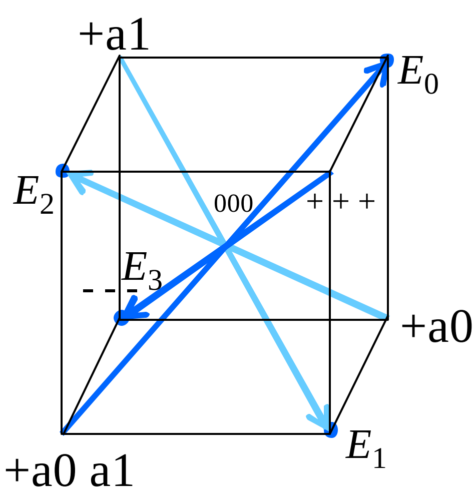
$R_k = -P_k$

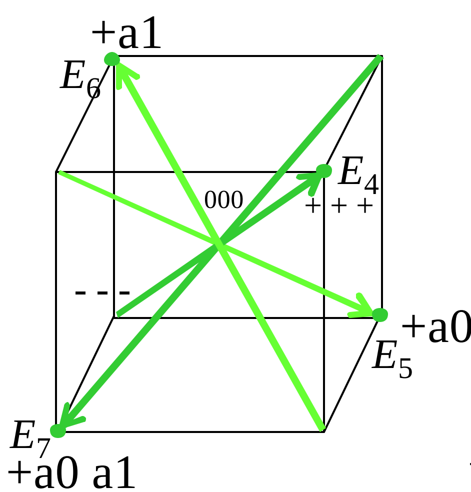
$E_k^2 = 1$

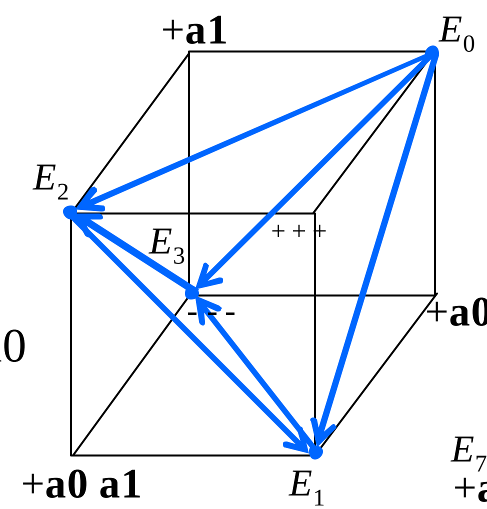
$E_k R_k = R_k$

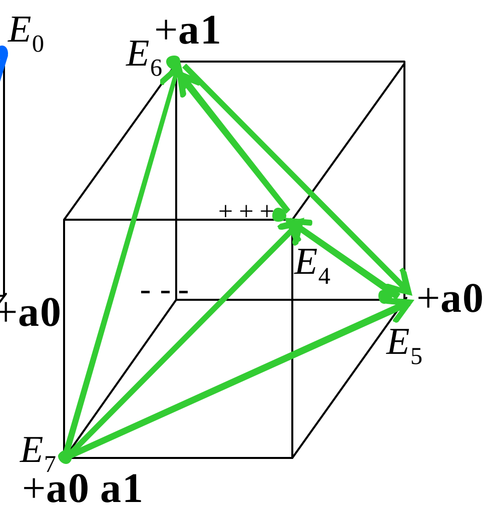
$P_k^2 = P_k$

Idempotent!!



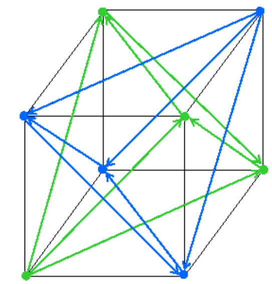






$E_k = \pm \mathbf{a0} \pm \mathbf{a1} \pm \mathbf{a0\ a1}$

$P_0 \bullet P_3 = P_1 \bullet P_2 = P_7 \bullet P_4 = P_6 \bullet P_5 = 0$



Qubits form Quantum Register $\boldsymbol{Q}_{\mathrm{q}}$

with A = (±**a0** ±**a1**), B = (±**b0** ±**b1**), C = (±**c0** ±**c1**)
then $A\ B\ C$ = (±**a0** ±**a1**)(±**b0** ±**b1**)(±**c0** ±**c1**) so

$A_+ B_+$ = (+**a0** +**a1**)(+**b0** +**b1**) = **a0 b0** + **a0 b1** + **a1 b0** + **a1 b1**

Geometric product replaces the tensor product ⊗

$\boldsymbol{Q}_{\mathrm{q}} = \boldsymbol{G}_{\mathrm{n=2q}}$

Row $_k$	State Combinations				Individual bivector products				Column Vector	
	a0	**a1**	**b0**	**b1**	**a0 b0**	**a0 b1**	**a1 b0**	**a1 b1**	$A_+ B_+$	$A_0 B_0$
R_0	–	–	–	–	+	+	+	+	+	0
R_3	–	–	+	+	–	–	–	–	–	0
R_5	–	+	–	+	+	–	–	+	0	–
R_6	–	+	+	–	–	+	+	–	0	+
R_9	+	–	–	+	–	+	+	–	0	+
R_{10}	+	–	+	–	+	–	–	+	0	–
R_{12}	+	+	–	–	–	–	–	–	–	0
R_{15}	+	+	+	+	+	+	+	+	+	0

State Count:

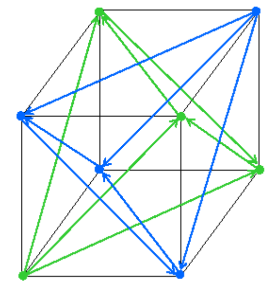
Total: $2^{2q} = 4^q$

Non-zero: 2^q

Zeros: $4^q - 2^q$

$A\ B\ C = 0$

$A_1 B_1 P_A P_B =$
a1 b1 = $\mathbf{S}_{11}$



Ebits: Bell/magic States and Operators

Separable: $A_0 B_0 (\mathbf{S}_A)(\mathbf{S}_B) = A_0 (\mathbf{S}_A) B_0 (\mathbf{S}_B) = A_+ B_+$

Non-Separable: $A_0 B_0 (\mathbf{S}_A + \mathbf{S}_B) = A_+ B_0 + A_0 B_+$ *Concurrent*!

$= -\mathbf{a0\ b0} + 0\ \mathbf{a0\ b1} + 0\ \mathbf{a1\ b0} + \mathbf{a1\ b1}$

$= -\mathbf{a0\ b0} + \mathbf{a1\ b1} = \mathbf{S}_{00} + \mathbf{S}_{11} = \boldsymbol{B}_0$

Row $_k$	State Combinations				Individual bivectors		Output column
	a0	**a1**	**b0**	**b1**	**–a0 b0**	**a1 b1**	
R_1	–	–	–	+	–	–	+
R_2	–	–	+	–	+	+	–
R_4	–	+	–	–	–	–	+
R_7	–	+	+	+	+	+	–
R_8	+	–	–	–	+	+	–
R_{11}	+	–	+	+	–	–	+
R_{13}	+	+	–	+	+	+	–
R_{14}	+	+	+	–	–	–	+

Valid states where exactly *one* qubit in superposition phase!!

$\boldsymbol{B} = (\mathbf{S}_A + \mathbf{S}_B)$

$\boldsymbol{B}_{i\pm1} = \pm\boldsymbol{B}_i \boldsymbol{B}$

$\boldsymbol{B}_0 = -\mathbf{S}_{00} + \mathbf{S}_{11} = \Phi^+$

$\boldsymbol{B}_1 = +\mathbf{S}_{01} + \mathbf{S}_{10} = \Psi^+$

$\boldsymbol{B}_2 = +\mathbf{S}_{00} - \mathbf{S}_{11} = \Phi^-$

$\boldsymbol{B}_3 = -\mathbf{S}_{01} - \mathbf{S}_{10} = \Psi^-$

$\boldsymbol{M} = (\mathbf{S}_A - \mathbf{S}_B)$

$\boldsymbol{M}_{i\pm1} = \pm\boldsymbol{M}_i \boldsymbol{M}$

$\boldsymbol{M}_0 = +\mathbf{S}_{01} - \mathbf{S}_{10}$

$\boldsymbol{M}_1 = -\mathbf{S}_{00} - \mathbf{S}_{11}$

$\boldsymbol{M}_2 = -\mathbf{S}_{01} + \mathbf{S}_{10}$

$\boldsymbol{M}_3 = +\mathbf{S}_{00} + \mathbf{S}_{11}$

$\boldsymbol{M}_3 = \boldsymbol{B}_2 (\mathbf{S}_{01} + \mathbf{S}_{10})$

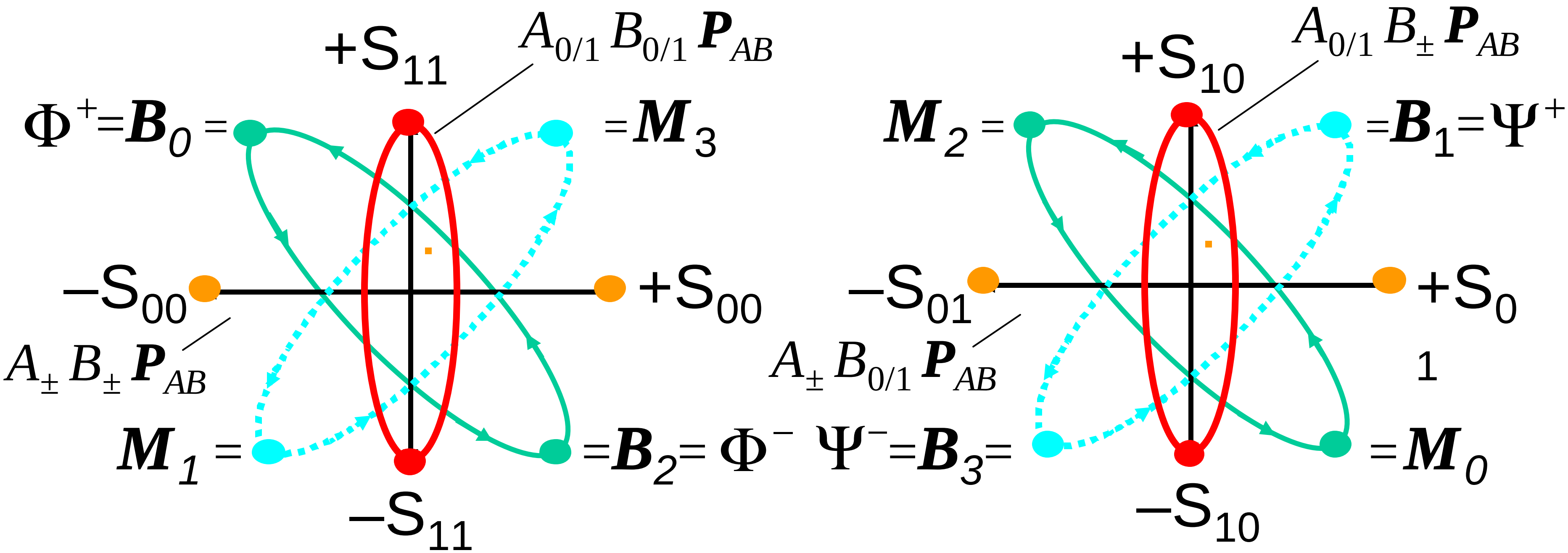
B & ***M*** are *Singular*!

Interesting Facts about Ebits

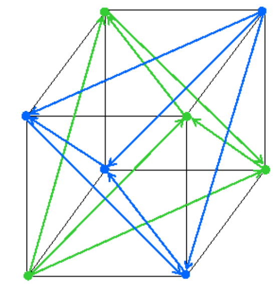
$\boldsymbol{B}^2 = \boldsymbol{I}^-$ and $\boldsymbol{M}^2 = \boldsymbol{I}^-$
and $\sqrt{\boldsymbol{B}} = \boldsymbol{B} + \boldsymbol{I}^-$

$-\boldsymbol{P}_A \boldsymbol{P}_B = \boldsymbol{B} - (1 + \mathbf{S}_A \mathbf{S}_B) = \boldsymbol{B} + \boldsymbol{I}^+$

$A\,B = \boldsymbol{B}_i + \boldsymbol{M}_j$ but
$\boldsymbol{B}_i \boldsymbol{M} = \boldsymbol{M}_i \boldsymbol{B} = 0$,
so $A\,B\,\boldsymbol{B} = \boldsymbol{B}_{i+1} + 0$



B and ***M*** are *valid for* $\boldsymbol{Q}_{q>2}$ *as* $(\mathbf{S}_A \pm \mathbf{S}_B \pm \mathbf{S}_C \pm \ldots)$



Cnot, Cspin and Toffoli Operators

For $\boldsymbol{Q}_2$ with qubits A and B, where A is the control:

$\mathrm{CNot}_{AB} = A_0 = (\mathbf{a0} - \mathbf{a1})$ where $(A_0)^2 = -1$

$\mathrm{Cspin}_{AB} = \sqrt{\mathrm{CNot}} = (-1 + A_0) = (-1 + \mathbf{a0} - \mathbf{a1})$

For $\boldsymbol{Q}_3$: qubits A, B & D where A & B are controls:

$\mathrm{Tof}_{AB} = \mathrm{CNot}_{AD} + \mathrm{CNot}_{BD} = A_1 + B_0$ (***concurrent***!)

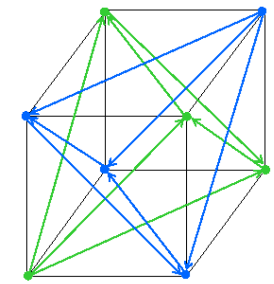
$= -\,\mathbf{a0} + \mathbf{a1} + \mathbf{b0} - \mathbf{b1}$ where $(\mathrm{Tof}_{AB})^2 = 1$

Row$_k$	State Combinations						Active States	$A_0\,B_0\,D_0\,(TOF_{AB})$	
	a0	**a1**	**b0**	**b1**	**d0**	**d1**			
R_{21}	–	+	–	+	–	+	$A_1\,B_1$ & D_1	–	Inverted
R_{22}	–	+	–	+	+	–	$A_1\,B_1$ & D_0	+	
R_{41}	+	–	+	–	–	+	$A_0\,B_0$ & D_1	+	Identity
R_{42}	+	–	+	–	+	–	$A_0\,B_0$ & D_0	–	

Also for $\boldsymbol{Q}_q$

$P_k^{2q} = P_k$

$E_k^{x} = 1$	
q	x
1	2
2	6
3	80
4	???



Conclusions

- The Quantum Geometric Algebra approach appears to simply and elegantly define many of the properties of quantum computing.
- This work was facilitated tremendously by the use of custom tools that automatically maintained the GA anticommutative and topological rules in an algebraic fashion.
- Many thanks to Mike Manthey for all his inspiration and support on my PhD effort.
- Many questions and much work still remains.