Information and available work in the perturbed baker's map

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Abstract

An analysis of the baker's map, a prototype for chaotic Hamiltonian evolution, shows evidence of a new kind of hypersensitivity to perturbations, which is based on Landauer's principle and on the notion of available work. The Landauer erasure cost of the algorithmic information needed for a detailed tracking of the time evolution of a perturbed state leads to a rapid decrease of free energy, making it energetically favorable to coarse grain and to put up with the resulting standard increase in entropy. This provides an explanation of the Second Law. A similar approach can be taken in the quantum case, suggesting a new signature of quantum chaos.

1 Introduction

Information theory and physics are connected through the concept of entropy. Ordinary entropy measures the degree to which incomplete knowledge about a physical system reduces the ability to extract work from that system; it thus appears as a negative contribution to free energy. Computation and physics are connected through Landauer's principle [1, 2], which specifies the unavoidable energy cost $k_B T \ln 2$ connected with the erasure of a bit of information. As a consequence of Landauer's principle, the information (quantified by algorithmic information [3]) needed to give a complete description of the system state also reduces the amount of available work and thus should be added as a further negative contribution to free energy [4, 5].

This paper is motivated by the general question [6] of how available work decreases during chaotic Hamiltonian evolution subjected to energy-conserving perturbations. Section 2 focuses on the classical baker's transformation [7] (see Fig. 1), a prototype of a chaotic area-conserving map, while Section 3 contains a short discussion of the corresponding quantum case. General considerations [6] suggest that the results found here are not limited to the specific example of the baker's map. Moreover, a generalization to the large class of systems that can be modeled by symbolic dynamics [8] seems to be possible. A more detailed discussion is given in Ref. [9].

2 Classical chaos

It is well known that, for mixing systems, a suitably defined coarse-grained entropy increases with time, approaching some equilibrium value. In Ref. [10], it is shown explicitly that the coarse-grained probability density for the baker's transformation approaches a constant in the long-time limit. It is tempting to regard this result as a proof of the Second Law. The coarse graining, however, appears as an *ad hoc* assumption. We show that the presence of perturbations in any realistic system provides a justification for coarse graining.

It is intuitively obvious that it would be crazy to try to follow the detailed behavior of a perturbed chaotic system. We quantify this intuition precisely by showing that coarse graining is the work-efficient strategy in the presence of perturbations. By this we mean that keeping track of the perturbed phase-space pattern leads to a much greater reduction in free energy than averaging over the small-scale perturbations. The reason is that the perturbation accesses algorithmically complex patterns of the sort discussed in Refs. [6, 11]. The information needed to specify a typical such pattern is so enormous that its Landauer erasure cost far outweighs the entropy increase due to averaging.

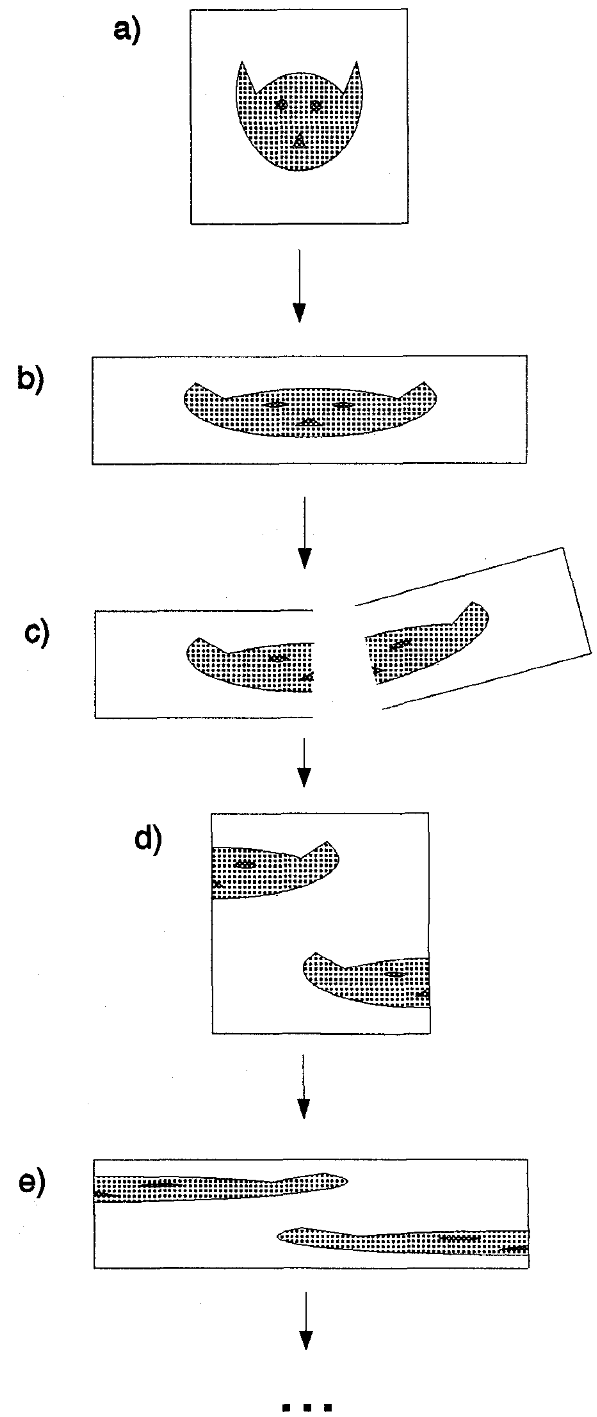

Figure 1: The baker's map transforms the cat in a) into its image in d). After the original pattern in a) has been stretched and cut, the image in d) is obtained by stacking the right part on top of the left part. Repeated application of the map leads to a pattern consisting of many horizontal stripes. There is an obvious analogy to the way a baker kneads dough.

The concept of algorithmic information was introduced into physics as a way to characterize chaos [8, 12]. This characterization focuses on the algorithmic information needed to calculate a system trajectory. For discrete chaotic maps, the number of initial-condition digits needed to specify a trajectory to a given accuracy increases linearly with the number of steps one wishes to predict. Because we are interested in the question of available work, our approach is fundamentally different. We start with an initial condition given to some finite accuracy, corresponding to a finite initial area in phase space. Under the chaotic time evolution, this area is repeatedly stretched and folded, thereby evolving into an apparently complicated pattern. Since knowledge of the pattern, no matter how much stretched and folded, permits extraction of all the work available initially, we ask for the algorithmic information needed to specify this pattern after n steps. By describing the system in terms of its symbolic dynamics [8], it can be shown easily that—given the background information to describe the system and the initial condition—only the number n is needed [5] to specify the evolved pattern completely, requiring $\Delta I(n) \simeq \log n$ bits of information. This information is negligible, in sharp contrast with the diverging amount of information needed to specify a single trajectory. This means that the free-energy cost of keeping track of the evolved pattern grows very slowly. Unperturbed evolution of an algorithmically simple initial state does not lead to algorithmically complex states, at least for reasonable values of n. It is therefore possible in principle to retain the ability to extract work by keeping track of the evolving pattern.

These ideas apply directly to the baker's map, whose action on a pattern in phase space is illustrated in Fig. 1. Figure 2 further illustrates the unperturbed baker's map and, in addition, gives an example of a perturbed baker's map that we discuss below. In Fig. 2, phase-space patterns are given both as areas in the unit square and in symbolic representation. Each point in phase space is represented by a symbolic string $s = \cdots s_{-2}s_{-1}s_0.s_1s_2\cdots$ where $s_k = 0$ or 1. The string s is identified with a point (q,p) in the unit square by setting $q = \sum_{k=1}^{\infty} s_k 2^{-k}$ and $p = \sum_{k=0}^{\infty} s_{-k}2^{-k-1}$. To specify *sets* of points, we write $s_k = x$ if s_k is undetermined; the set $\{(q,p); 0 \leq q,p \leq 1/2\}$ is thus denoted by $\cdots xx0.0xx\cdots$ or simply by $xx0.0xx$—i. e., $s_0 = s_1 = 0$ and $s_k = x$ for $k \notin \{0,1\}$. We identify a set of strings with a uniform probability distribution on the union of the areas represented by the strings. The action of the baker's map on a symbolic string is given by the shift map U de-

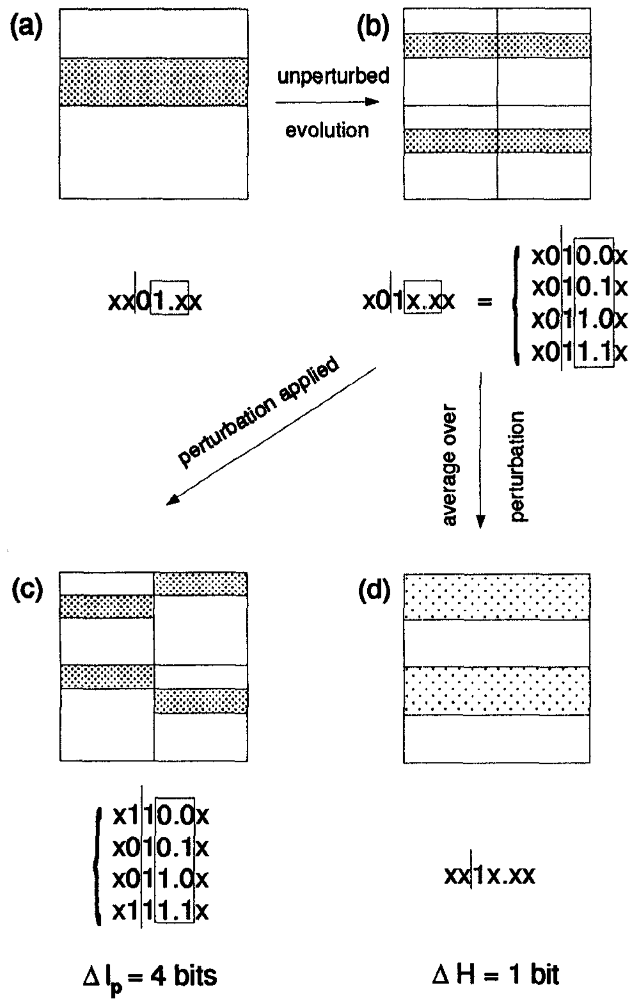

Figure 2: An example of the action of the perturbed baker's map on the initial condition shown in (a). In (b), the pattern that results from application of the unperturbed baker's map is split into four subpatterns determined by the four perturbation cells. In the symbolic representation, these perturbation cells are distinguished by two decision digits, here enclosed in a box. In (c), a perturbation map is applied to each subpattern independently, affecting only the digits in the perturbation region to the left of the vertical line. The information needed to specify the perturbed pattern, given the initial pattern and the number of steps, is $\Delta I_p = 4$ bits. Averaging over all possible perturbation maps leads to the coarse-grained pattern in (d), with an entropy increase of $\Delta H = 1$ bit.

fined by $(Us)_k = s_{k+1}$.

The slow, logarithmic growth of algorithmic information for the unperturbed baker's map changes dramatically if one allows for perturbations. We have defined a perturbed version of the baker's map, for which the algorithmic information needed to keep track of a perturbed pattern in fine-grained detail can be calculated exactly. This information has to be compared to the increase in entropy, ΔH, that results from averaging over the perturbations. We find that ΔH increases at most linearly with the number of steps n, whereas the information needed to keep track of the pattern, besides being always larger than ΔH, has a regime where it increases exponentially with n.

In Fig. 2, a perturbed time step is applied to the initial pattern $xx01.xx$ shown in (a). The region lying just to the left of the digit 0, as indicated by the vertical line, we call the *perturbation region.* In (b), the pattern $x01x.xx$ which results from application of the unperturbed baker's map is split into four subpatterns that partition the unit square into four congruent *perturbation cells.* In the symbolic representation, these perturbation cells are distinguished by two *decision digits*, here enclosed in a box. In (c), a random *perturbation map* is applied to each subpattern independently, affecting only the digits in the perturbation region to the left of the vertical line. Since, at this first step, only one initial-condition digit of each subpattern is located in the perturbation region, there are only two possible perturbation maps for each perturbation cell, the identity map and the *switch map* that interchanges 0 and 1.

In the following, we let $\Delta I_p(n)$ be the additional (conditional) algorithmic information needed to specify a *typical* perturbed pattern, given the background information and the number of steps n. Since the patterns are generated by a random mechanism that leads to equally likely alternatives, ΔI_p can be determined simply by counting [6, 13]: if N_p is the number of equally likely patterns, the information needed to specify a typical pattern is $\Delta I_p \simeq \log N_p$. In the example of Fig. (c), there are $N_p = 16$ patterns—two maps for each of four perturbation cells—leading to $\Delta I_p = 4$ bits. Averaging over all possible perturbation maps leads to the coarse-grained pattern $xx1x.x$ in (d), with an entropy increase of $\Delta H = 1$ bit.

In Fig. 3, a similar example is followed through a larger number of steps. The left column shows the unperturbed evolution of the initial ($n = 0$) pattern; since the phase-space area of the pattern remains constant, there is no change in entropy, $\Delta H = 0$. Given the background information, the evolved pat-

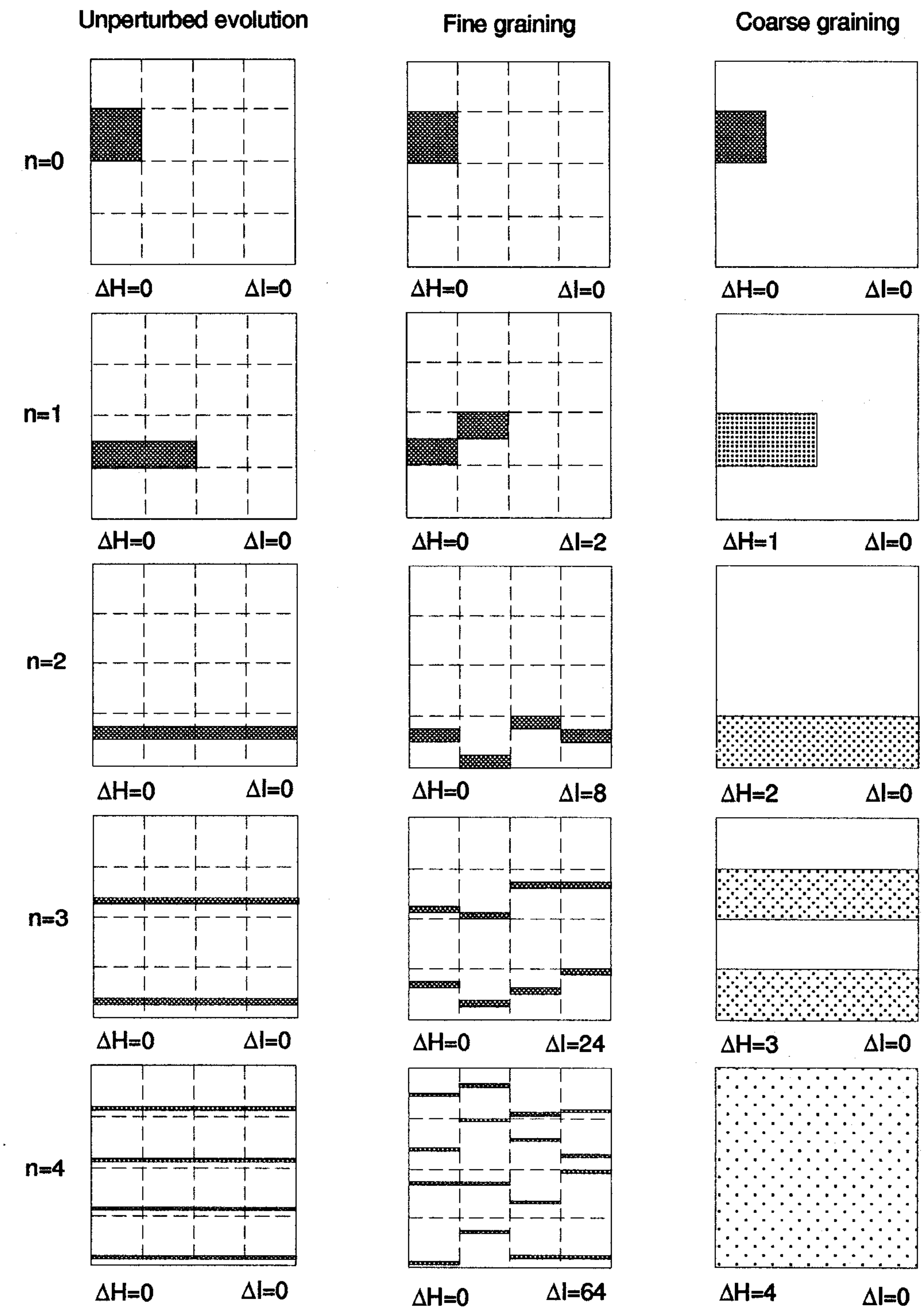

Figure 3: Unperturbed evolution of a phase-space pattern versus two strategies for following the perturbed evolution. The amount of algorithmic information ΔI needed for fine graining is much larger than the standard entropy increase ΔH that results from coarse graining.

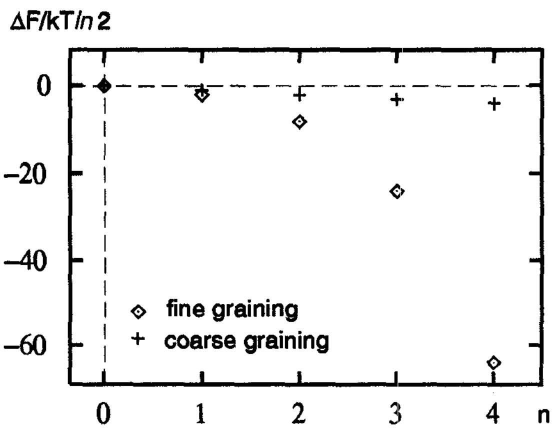

Figure 4: Free energy change $\Delta F = k_B T \ln 2(\Delta H + \Delta I)$ versus number of steps n for the perturbed evolution shown in Fig. 3. The ability to extract work is preserved much better in the case of coarse graining.

tern is completely described by the amount of information, $\Delta I = \log n$, needed to specify the number of steps n. Neglecting the small quantity $\log n$, we can write $\Delta I = 0$; i. e., almost no information is needed to follow the unperturbed evolution of the pattern.

The second column of Fig. 3 shows a particular realization of a perturbed evolution in fine-grained detail. Just as for the unperturbed evolution, there is no change in the area of the pattern, and therefore $\Delta H = 0$. By contrast, the information needed to specify the pattern increases exponentially: at each step, there is a doubling of the number of perturbation cells in which the position of the pattern has to be specified. In addition, the position of the partial pattern in each cell has to be given with increasing accuracy if the fine-grained structure of the perturbed pattern is to remain visible. As an example, consider $n = 3$: we find that $\Delta I = 24$ because the pattern extends over 8 perturbation cells and, in each cell, three bits are needed to locate the pattern.

In the rightmost column of Fig. 3, a coarse-grained view is taken by averaging over the possible perturbations. The phase-space area doubles with each step, causing ΔH to increase linearly. As in the unperturbed case, we find $\Delta I = 0$, meaning that the coarse-grained pattern is completely specified by the number of steps n.

The results of the example in Fig. 3 are summarized in Fig. 4, where the change in free energy, $\Delta F = k_B T \ln 2(\Delta H + \Delta I)$ is plotted versus the number of steps for both fine graining and coarse graining. It is obvious that ΔF decreases much more slowly if the strategy of coarse graining is chosen. This is our key result for classical chaotic systems: In the presence of perturbations, coarse graining is the better strategy to preserve the ability to extract work; fine graining has a much higher free-energy cost due to the large amount of information needed to describe a typical perturbed pattern.

3 Quantum chaos

In this section, we describe briefly results we obtained recently for a quantized version of the baker's map. Here, we are interested in following the time evolution of a pure state in Hilbert space under the action of a time-evolution operator that is perturbed randomly. As in the classical case, we compare the algorithmic information needed to track the perturbed state in detail with the standard increase in von Neumann entropy when an average over the perturbation is performed.

There is no unique way to quantize a classical map. We use the quantum baker's map first introduced by Balazs and Voros [14] and presented in a more symmetric form by Saraceno [15]. We have defined a set of perturbation operators that, in the classical limit, resemble closely the type of perturbation used in the previous section.

We have chosen a realization of the quantum baker's map in $d = 16$-dimensional Hilbert space. Starting with an initial pure state, we apply at each step randomly one of two possible perturbation operators. After $n = 15$ steps, there are 2^{15} possible trajectories. Using a computer, we have checked that all these trajectories lead to distinct Hilbert-space vectors, where we call two vectors distinct if their scalar product is smaller than 0.99, corresponding to a separation in Hilbert-space angle bigger than about 8°. Applying the same counting argument as in the previous section, we find that the algorithmic information needed to specify a typical perturbed vector after $n = 15$ steps is $\Delta I = 15$, which is of the order of the dimension of Hilbert space, $d = 16$. This is much larger than the maximum value of the von Neumann entropy, $\Delta H = \log d = 4$. The quantum baker's map thus shows the same sensitivity to perturbations as the classical baker's map.

4 Summary and conclusions

For the perturbed baker's map, the information needed to keep track of the evolving pattern far exceeds the entropy that results from averaging over the perturbation. This key result expresses a hypersensitivity to perturbations. We expect all chaotic systems with positive KS entropy (or metric entropy) to display a similar hypersensitivity to perturbations [6]. In contrast, classical regular systems should not display such hypersensitivity.

The quantum baker's map displays a hypersensitivity to perturbations similar to the classical baker's map, thus suggesting a new connection between chaos and quantum mechanics. There is evidence [6] that this connection constitutes a general way to define quantum chaos which is directly relevant to statistical physics.

It is worth reiterating the physical meaning of our key result. Whenever $\Delta I_p \gg \Delta H$, averaging over the perturbation, which amounts to a coarse graining matched to the strength of the perturbation, is a far better strategy for preserving available work than is keeping track of the perturbed system state in fine-grained detail. This is a compelling justification for coarse graining in the presence of a perturbation and for the accompanying increase in entropy. There is a way around this conclusion: the excess information needed to specify the fine-grained pattern can be used to extract an equivalent amount of work from the perturbing system. Given a split between a system of interest and its surroundings, however, we have provided an explanation of the Second Law.

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