The Energy Content of Knowledge

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Abstract

This paper explores the implications of a Maxwell's demon successfully applying a priori knowledge about the motion of particles to a thermodynamic system. The assumption that the knowledge can be recorded symbolically is used to calculate the number of bits required to represent the knowledge needed for placing particles of a gas into a maximum work configuration. The amount of work is divided by the number of knowledge encoding bits to derive the average work potential per bit. The quantum mechanics correspondence principle and a conservation of energy analysis are used to argue that each knowledge bit must represent a quantity of energy, and the sum total of the knowledge energy is the quantity of work done by the system.

1 Introduction

Maxwell proposed the existence of a creature which could open and shut a small door located in a wall separating two chambers with such agility as to allow only atoms of higher kinetic energy to pass through the door in one direction, and those of lower energy to pass in the other (Figure 1). According to the classical kinetic model of an ideal gas, such finely tuned control of a door could be used to force heat from lower to higher temperatures in violation of the second law of thermodynamics. Lord Kelvin later named Maxwell's proposed creature "Maxwell's demon".

The implications of such a possibility have been a source of curiosity, criticism, and controversy since the day the editor first saw it in Maxwell's thermodynamics book in 1867. Perhaps the demonic label is justified. Many of the salient papers which range in year of authorship from 1874 to 1988 are collected in [1]. These papers deal with the important limitations of measurement and entropy in relationship with the preservation of the second law. The central issues surrounding the problem concern obtaining and applying knowledge - namely the knowledge of the location and momentum of the gas particles.

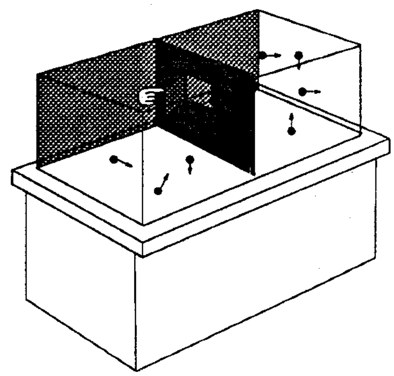

Figure 1: Maxwell's Demon Model

This paper focuses on finding the value of the demon's knowledge. In this variation on Maxwell's Gedanken experiment, the source of the demon's knowledge is not scrutinized - we assume the knowledge exists a priori. In Thompson's parlance, we are assuming that this creature of another realm need not be affected by the components of a thermodynamic system, and that its chthonian knowledge is given.

2 Size of the Required Knowledge

This section develops a relationship between the information theoretic quantity of knowledge that is measured in 'bits' and the quantity of kinetic energy that is measured in Joules. The argument is based on the pressure version of the Maxwell's demon model that was proposed by Leo Szilard [2]. The machine is held at constant temperature by a large reservoir. The demon has knowledge of when to open or shut the door, so given an arbitrarily long time the gas will approach an organization that can deliver the maximum amount of energy, i.e. all of the gas particles will be located in one chamber. The minimum number of bits required for a nonredundant encoding of the demon's knowledge divided by the deliverable energy will give the average energy potential of a knowledge bit.

The demon could use different approaches to solve his problem. For one he could just open and shut the door at the correct times. The demon might have an internal clock and a list of times associated with actions. Such directions would be encoded as, at time 1 open the door, at time 2 close the door, at time 5 open the door, ...etc. This could be simplified into an array of bits, one bit per time slot.

In another approach he could leave the door open and wait until all of the gas particles are on the correct side by chance, and then shut the door. According to this approach, the demon would have to know how long to wait. This would require encoding the wait time with a resolution sufficient that the quantizing error would not be larger than the period when the gas particles remained in the optimum configuration. The demon's knowledge would be coded with a single number.

Independent of the demon's methodology, the minimum knowledge encoding length will asymptotically approach the information content of a message containing the destination state. This follows from the fact that the demon's knowledge is a message to the system on how to obtain the destination state. According to Hartley [3] and Shannon [4][5], the information content of this message is related to the probability of its occurring (1). This equation can be justified upon purely symbolic arguments in communications theory, independent of thermodynamics.

$$K_{min} = I_d = \log_2 \frac{1}{P_d} \tag{1}$$

K_{min} = minimum knowledge encoding length
I_d = information content
P_d = destination state probability

The Szilard ensemble gas model is shown in figure 2. The probability of the destination state occurring in the ensemble is inversely proportional to the number of possible gas states weighted by the probability of being in each state. In the ith one particle Szilard component the probability of being in the destination chamber is P_{di} (2). For the ensemble, the probability of being in the destination state is the coincidental probability (3).

$$P_{di} = \frac{V_d}{V_t} \tag{2}$$

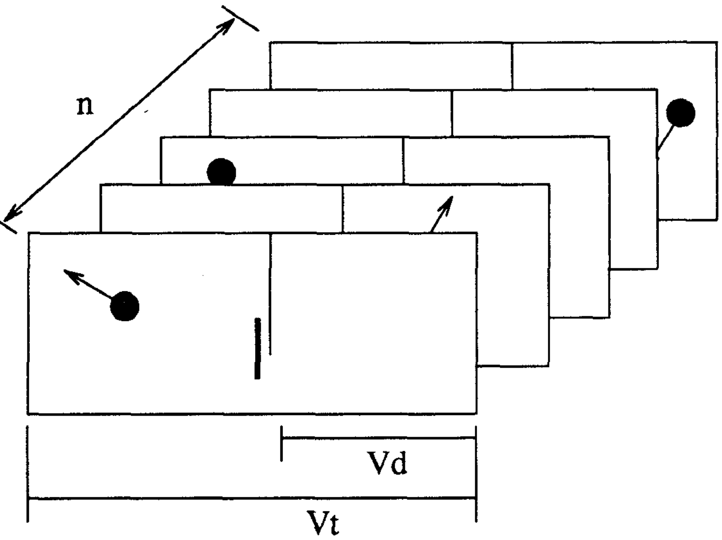

Figure 2: Ensemble Model

P_{di} = probability of chamber i being in state d
V_d = volume of destination chamber
V_t = total volume

$$P_d = \prod_i P_{di} = \left(\frac{V_d}{V_t}\right)^n \tag{3}$$

P_d = probability of gas being in state d
n = number of particles

By substituting the probability of being in the destination state (3) into the measure of information (1) the minimum length knowledge encoding can be stated in terms of the number of particles in the system and the geometry of the chamber, V_t/V_d:

$$K_{min} = n \log_2 \frac{V_t}{V_d} \tag{4}$$

This result shows that the minimum encoding is one bit per particle times a geometry determined value and the constant $\log_2$. However, this does not determine the demon's memory requirements since he might learn the knowledge in pieces while rewriting the same memory over and over.

3 Work Obtainable From the Knowledge

After the demon finishes sorting, the particles are all in one chamber, so the system can undergo isothermal expansion to do work. The formula for isothermal expansion of an ideal gas is:

$$W = nkT \ln \frac{V_t}{V_d} \quad (5)$$

The average amount of work per bit (6), W_b, is derived by dividing (5) by (4). The number of particles n, and the geometry dependent value cancels. This leaves energy per bit as a constant. This constant is composed of Boltzman's constant, the ambient temperature, and $\ln 2$.

$$W_b = kT \ln 2 \quad (6)$$

For a real system at room temperature (298 deg K) the ideal work potential of a bit is:

$$W_b \doteq 2.85 \cdot 10^{-21} J/b \quad (7)$$

It is interesting to note that base e, not base 2, is the most efficient number base for encoding information [6]. When knowledge is stored in cells of e base, equation (6) loses the $\ln 2$, and there are an equal number of bits and particles, and equal amount of average energy carried by each:

$$E_{\text{bmin}} = kT \quad (8)$$

This section shows a correspondence between knowledge needed to activate a thermodynamic system, and the amount of energy which is activated. The following sections will argue that the work comes from energy that was transferred from the bits.

4 Momentum Shift

A quantum mechanical analysis of a one dimensional adiabatic system will show the principles involved. The ψ function is the solution to the Schrödinger wave equation (9). According to the model for the 'waiting' demon, the ψ position function is scaled when the demon places the divider. The original system state is described by $\psi_1(x)$ and the final state is described by $\psi_2(x)$. This argument assumes that placing the divider slowly does not cause a significant perturbation.

$$i\hbar\frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = 0 \quad (9)$$

Since the energy of a particle in ψ_2 is a free variable, Schrödinger's wave equation contains no references to absolute distances relative to the large chamber distances, and the shape of the chamber after the divider is placed is geometrically similar to the original shape, it follows that ψ_2 differs from ψ_1 by a scaling factor. All ψ^2 functions are probability distributions, hence:

$$\int_{-\infty}^{+\infty} \psi_2^2(x_2)dx_2 = \int_{-\infty}^{+\infty} \psi_1^2(x)dx = 1 \quad (10)$$

The following co-ordinate transform converts the gas chamber in the initial configuration to the destination configuration:

$$x_2 = \frac{1}{a}x \quad (11)$$

$$dx_2 = \frac{1}{a}dx \quad (12)$$

Where a is defined to be V_t/V_d, a number which will always be greater than one when the gas is compressed. Applying the transform to the left hand side of equation (10) yields:

$$\int_{-\infty}^{+\infty} \psi_2^2(\frac{1}{a}x)\frac{1}{a}dx = \int_{-\infty}^{+\infty} \psi_1^2(x)dx = 1 \quad (13)$$

The following scaling relationship between the two ψ functions satisfies the equation (13) constraint:

$$\psi_2(x_2) = \sqrt{a}\psi_1(x) \quad (14)$$

The corresponding scaling of the complex conjugate and the first derivative:

$$\psi_2^*(x_2) = \sqrt{a}\psi_1^*(x) \quad (15)$$

$$\frac{d\psi_2}{dx_2} = \sqrt{a}\frac{d\psi_1}{dx}\frac{dx}{dx_2} = a^{\frac{3}{2}}\frac{d\psi_1}{dx} \quad (16)$$

The well known equation for the expected value of momentum applied to the system before the divider is inserted:

$$< p_1 >= \int \psi_1^*(x)\frac{\hbar}{i}\frac{d\psi_1(x)}{dx}dx \quad (17)$$

The steady state equation for the system after the divider is inserted (i.e. after the demon shuts the door):

$$< p_2 >= \int \psi_2^*(x_2)\frac{\hbar}{i}\frac{d\psi_2(x_2)}{dx_2}dx_2 \quad (18)$$

This equation is transformed back to the 'x' coordinate system by directly substituting the relationships given in (14) (15) (16) to yield:

$$< p_2 >= \int a^{\frac{1}{2}}\psi_1^*(x)\frac{\hbar}{i}a^{\frac{3}{2}}\frac{d\psi_1(x)}{dx}a^{-1}dx \quad (19)$$

Which simplifies to:

$$< p_2 >= a \int \psi_1^*(x) \frac{\hbar}{i} \frac{d\psi_1(x)}{dx} dx \tag{20}$$

This relationship can be described with equation (17) scaled by a:

$$< p_2 >= a < p_1 > \tag{21}$$

Scaling the wave function causes the expected value of the momentum to shift. This same result can be derived from the Fourier transform relationship between the position and momentum distributions, $w(\frac{1}{a}x) \Leftrightarrow aW(ak)$.

Since the energy change in the system is proportional to the square of the momentum:

$$E_d - E_t = E_t(a^2 - 1) \tag{22}$$

5 Classical Adiabatic Compression

We now calculate the energy change of a gas caused by adiabatic mechanical compression as a point of reference for the previous quantum mechanical derivation. A significant difference between this calculation and the previous quantum mechanical one is the presence of terms specific to the mechanical nature of the source of compression. For the case of a one dimensional gas, i.e. a gas in a differential filament, it follows that the pressure on the piston is:

$$< dF >= \frac{1}{T} \int_{t1}^{t2} \frac{1}{2} N \frac{v\, dA}{V} \cdot 2dm v dt \tag{23}$$

Where dm is the differential mass of a particle in the filament, dA is the differential cross sectional area, N is the number of particles, and $< dF >$ is the expected value of differential force. Since all the variables are assumed constant with respect to time, except for particle velocity:

$$PV = \frac{dF}{dA} V = Nm < v^2 > \tag{24}$$

Since $N \cdot \frac{1}{2} m < v^2 >$ is the internal energy of the ideal gas:

$$E = 2PV \tag{25}$$

The work done by the gas when moving a piston is:

$$dW = PdV \tag{26}$$

Since this is an adiabatic system, the only energy increase can come from the work placed into the system (which is negative from work done by the system), hence:

$$\begin{aligned} dE &= -dW \\ d(\frac{1}{2}PV) &= -PdV \\ 3PdV + VdP &= 0 \end{aligned}$$

$$PV^3 = C \tag{27}$$

Solving for P and substituting back into the energy expression yields:

$$E = \frac{2C}{V^2} \tag{28}$$

It follows from equation (28), and from the transform (11) where x is taken as V for the one dimensional gas, that:

$$E_d - E_t = E_t(a^2 - 1) \tag{29}$$

The solution for the classic system is the same as that for the quantum mechanical system. The conclusion that energy is added to the system when the wave function is scaled follows more simply from the correspondence principle, and the observation that there is no variable in the wave equation describing the source of the change. Constriction caused by a demon, or by a moving piston results in the same modification to ψ, and therefore the same modification to the momentum, and to the energy. Therefore we conclude:

> *Independent of the source of boundary changes around an QM gas (i.e. changes to the the distribution of ψ domain), the energy changes are consistent with the same boundary changes affected mechanically.*

As applied to Maxwell's Demon, we are claiming that division of a quantum mechanical system of particles in such a way that it is known that all of the particles in the system are located in a smaller volume afterwards requires placing energy into the system in the same quantity as would be done by mechanically compressing the gas into the smaller volume.

6 Conservation of System Energy

The machine undergoes two steps before it resumes its initial thermodynamic state. First there is a quasi-static isothermal compression performed by the Demon as he stuffs the molecules through the door. Second there is quasi-static isothermal expansion as work is removed from the system. Although the machine resumes its initial thermodynamic state upon completion of a cycle, it is very unlikely that all of the particles will be in the same locations with the same momentum. Even if the bits in the plan could have been copied, they no longer constitute knowledge.

In the first machine step heat is liberated into the reservoir. This follows from the correspondence principle since the wave function is scaled. The quantity of heat liberated is the same that would be liberated if a piston had physically isothermally compressed the gas:

$$Q = nkT \ln \frac{V_t}{V_d} \tag{30}$$

In the second machine step, isothermal expansion, heat is absorbed by the system. The quantity of heat is again (30), so there has been a net heat transfer of zero. However, as discussed in the first section, work was done (5). For conservation of energy to hold, there must have been an influx of energy. The gas cylinder and temperature reservoir were assumed to be completely isolated, and the gas cylinder and reservoir were returned to their initial states, so the energy must have come from the demon.

The mechanisms employed by the demon to convey the knowledge are also isolated from the outside. For instance, one proposed demon methodology was to simply wait, this requires little interaction with the gas, and no interaction with the outside. The only interface is the knowledge passed in by placing the divider. This knowledge was lost when energy appeared in the system. Also, according to the conservation of energy analysis, the magnitude of unexplained energy per bit is equal to the work potential of a bit, so conservation holds when this quantity of energy is assigned to a bit:

$$E_b = kT \ln 2 \tag{31}$$

7 Conclusions

In sections 2 and 3 we concluded there is a constant amount of energy activated per bit of knowledge possessed by Maxwell's demon. However, the energy source was was not located. The activation could have been interpreted as pure negative entropy, or perhaps as the source of energy. Since the demon does not physically touch the moving particles, the most plausible explanation appears to be that the knowledge is a source of negative entropy and that heat pours into the system to provide the work energy [7].

However, in sections 4 and 5 we showed that scaling the ψ function results in an increase in internal energy from the view point of the unscaled co-ordinates. The derivation was very general, specifying only a one dimensional quantum system. As a point of reference, the energy scaling was shown to be the same as caused by mechanical adiabatic compression of a one dimensional gas. Since the quantum mechanical equations were valid for any source of position distribution scaling, we concluded that compression caused by any means, including demons, must result in the same internal state changes.

The conservation of energy analysis in section 6, which used the correspondence principle developed in the quantum mechanical discussion, showed that the energy activated in the isothermal system must come from the knowledge bits. This result predicts that heat is released from a system as the demon successfully places the gas particles, and it is consumed in equal amounts during isothermal expansion.

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