The interplay between Gravitation and Information Theory

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In this article we review the surprising interconnection between gravity and information theory that emerged in the last twenty years. We focus attention on the recent developments in the area, namely, the problem of information transfer in space–time and the possible information-theoretic resolution of the cosmic censorship problem.

Space–time singularities

It is amazing how two as different areas of knowledge like information theory and gravitation became so intertwined in the last twenty years.

The origin of this close relationshipis rooted in the problem of space–time singularities. Physics is permeated by examples of singularities, for instance the electric field of a point-like (classical) charged particle. What makes space–time singularities so perverse is that these singularities *must* be part of space–time, as was proven through a series of beautiful theorems by Penrose and Hawking (see [1] for a complete review on the subject). In order to prove these theorems it is assumed that space–time is a smooth manifold, the positivity of the energy density, causal flux of matter, and some global causality condition (time orientability of the manifold, etc). Then it follows that the world-line of at least one observer cannot be extended beyond some point, which can be reached in a finite proper time. In other words, the observer must end his journey at some end-point, either in the past or in the future. Unfortunately, these theorems are unable to tell us the nature of these singularities. A further reason of worry about space–time singularities is that their very existence could threaten a holy principle of physics, predictability itself could be at stake! It is easy to understand why. Think of a singularity as a line that pierces a family of surfaces, each one representing a 'moment' of time. Attach to the world line of this singularity the corresponding light cones. Given the initial condition at one of these surfaces, the future development of any point inside these light cones is uncertain because the boundary condition is missing at the singular point.

The remedy to this problem was supposed to be the so-called *cosmic censorship hypothesis*, proposed by Penrose [2] in the sixties. Since the early thirties, the study of the gravitational collapse of stars seemed to indicate the formation of an event horizon, a surface which clothes all singularities in such a way that in its interior the future of all light cones points toward the singularity. What this means is that any signal emitted in this region cannot escape to the exterior, a black hole was formed. Thus, general relativity seems to defend itself against the threat of breakdown of predictability with its most powerful sword: causality.

Wheeler's demon

In the seventies, Wheeler realized that black holes could commit the perfect crime against the second law of thermodynamics: nobody could be aware of whether a system of finite entropy had ever been thrown inside a black hole or not. In analogy to Maxwell's demon, the gedanken experiment where the second law of thermodynamics could be violated by throwing thermodynamical systems inside black holes was called Wheeler's demon.

At the same epoch, it was found by Bardeen, Carter and Hawking [3] that if one replaces in the first and second laws of thermodynamics (up to some unknown constants) the entropy and the temperature of a thermodynamical system by the black hole event horizon and surface gravity there (essentially the force exerted by an observer at infinity by means of a rope to keep a second observer aloof close to the horizon), one obtains the four laws of black hole physics. Their derivation was based on purely gravitational premisses and they regarded the analogy as purely accidental, essentially because a black hole is not in any heat bath. However, Bekenstein did not share their view; he believed that this analogy was not fortuitous. It was known from the no-hair theorems [4,5] that no imprints of the initial conditions of the collapse other than the charge, angular momentum and mass can be observed after the black hole is formed. He then proposed that an entropy should be ascribed to the black hole, which represents all missing information about those conditions that are compatible with the end-point of the gravitational collapse [6]. He further assumed this entropy to be proportional to the event horizon area and estimated the proportionality constant, assuming that one bit of information is missed whenever an elementary particle crosses the horizon. This proposal could not explain how a black hole that is not in any thermal bath could be at some finite temperature. This paradoxical situation was only settled when Hawking [7] published his seminal work, based on field theoretic premisses, which confirmed this conjecture and

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showed that black holes are not black, they emit particles with a thermal spectrum as an ordinary black body.

If the second law of thermodynamics is to be generalized to include black holes as well as ordinary matter, then it must hold in every situation, in particular, whenever a thermodynamical system crosses the event horizon of a rotating black hole tangentially (no perpendicular momentum). It was soon realized by Bekenstein [8] that this will only be true if the infalling system satisfies the so-called 'entropy bound', a universal relation that constrains the entropy S *any* system could attain in terms of its proper size R and energy E:

$$S \leq \frac{2\pi R E}{\hbar c} . \qquad (1)$$

Knowing the close connection between entropy and information, this formula led Bekenstein [9] to rediscover a bound on the information transmission rate similar to the one proposed by Bremermann [10] in the early sixties:

$$H_{\max} \leq \frac{2\pi E}{\hbar} \log_2 e \text{ bits s}^{-1} . \qquad (2)$$

Since its early heuristic proposal in 1981, this bound has been supported by detailed numerical experiments [11] and by analytical arguments [12]- [14]. However, it was not until only a couple of years ago that rigorous proof of this conjecture, starting from the first principles of communication theory and statistical mechanics, became available [15]- [17]. It was then understood that the physical origin of the information/entropy bound rests on the indistinguishability of the information carriers (quanta of some field). It is surprising that black hole physics throws light in the realm of communication theory.

An unexpected connection

While trying to find an analytical proof for this bound, a new connection between information theory and gravity emerged. The broadband channel capacity formula for detectors submitted to thermal noise was obtained many years ago by Lebedeev and Levitin [18]. It quantifies the maximum amount of information that can be transmitted through a channel per unit time, in face of a noise that has a thermal behaviour. In this formula, the optimal information transmission rate is parametrized in terms of the signal's power P and the noise temperature T. On the other hand we knew from the seminal work of Unruh [19] that any detector uniformly accelerated with respect to the Minkowski vacuum feels as if it were immersed in a thermal bath at temperature $T_U = \hbar a/2\pi ck$, where a is the acceleration of the observer (proper frame) and k is Boltzmann's constant. In other words, an accelerated detector is submitted to thermal noise. Identifying $kT - \hbar a/2\pi ck$ (Unruh's thermal noise) in Lebedeev-Levitin's formula, we obtained the maximum amount of information that could be conveyed to the accelerated detector [17]. In the limit where the signal's power is small compared with the noise, we obtained

$$H_{\max} \approx (2\pi cP/\hbar a) \log_2 e \quad \text{bits s}^{-1} . \qquad (3)$$

The transition occurs at a characteristic power $P_c = 10^{-2}\hbar(a/c)^2$. Although for every-day accelerations this is a tiny power, for the acceleration typical of electrons in atoms (10^{25}.cm.s^{-2}), $P_c \approx 10^{12}$eV.s^{-1}, it is a huge figure. It thus may be that the transfer of information among elementary particles involved in natural processes is governed by the limiting form of eq. (3). Therefore acceleration will block the transmission of information, even between causally connected regions.

The problem of information detection by accelerated observers opened new vistas, because the physical source of Unruh's thermal noise is the inequivalence between the Minkowski vacuum and the one defined by the accelerated observer (Rindler) and because this inequivalence is a quite general feature of field theory in curved space-time. The reason is that the notion of vacuum is defined with respect to a time like Killing vector, a vector field that points to a (time) direction in space-time in which the geometry remains static. If such a vector field is lacking, particles are continuously created by the geometry, although the very notion of 'particle' is ambiguous in this case [20]. Actually, the concept of particle production can only be well defined if the space-time possesses regions that are asymptotically static in both the past and the future or, if we can work in the adiabatic approximation, assuming that the geometry changes in a rate that is not too fast compared with the frequency of the quanta produced.

Suppose now that someone sitting in a very distant region of the Universe wishes to send us optically some message. Then, owing to the cosmological process of particle production just mentioned, the received signal will differ in a stochastic way from the transmitted one. In other words, the gravitational field will be responsible for jamming the original signal. Even under these conditions, it is still possible to convey information. The object that quantifies the amount of information that could, in principle, be recovered from the output signal - even in the face of noise- is known in communication theory to be Shannon's mutual entropy [21]:

$$H(o;i) \equiv \sum_m p(m|n)p_i(n) \ln \frac{p(m|n)}{p_o(m)} . \qquad (4)$$

In this expression, $p_i(n)$ and $p_o(m)$ stand for the probabilities that n quanta were emitted and that m quanta were detected, respectively. The amount of noise present in the signal enters into this definition through the conditional probability $p(m|n)$ that m quanta were detected *knowing* that n were originally emitted. In order to obtain the regime where the transmission of information is optimized, given that $\langle m \rangle$ quanta are detected by the observer, we have to maximize the above expression with respect to either $p_i(n)$ or $p_o(m)$ (these two quantities are not independent, they are related via Jaynes identity). Let $H_{\max}$ be the corresponding channel capacity (in bits per pulse). The fundamental result proved by Shannon is that, via some clever coding of the message, the regime where information transmission is optimized can be reached, but never exceeded, *any information sent in excess of $H_{\max}$ will inevitably be degraded by noise* [21]. In other words, $H_{\max}$ represents the maximum amount of information that, in principle, could ever be conveyed between the emitter and the observer. The maximization of the conditional probability yields the following result [22]:

$$H_{\max} = \alpha + \beta \langle m \rangle . \tag{5}$$

In the above equation, α and β are Lagrange multipliers introduced to enforce the normalization of probability and the knowledge of the mean number of quanta. They satisfy the conditions [22]

$$\alpha = -\ln \sum_m e^{-(\beta m + B(m))} \tag{6}$$

and,

$$\langle m \rangle = -\frac{\partial \alpha}{\partial \beta} . \tag{7}$$

were

$$\sum_m B(m) p(m|n) = -\sum_m p(m|n) \ln p(m|n) . \tag{8}$$

The piece that is still missing is to specify the noise produced by the gravitational field, i.e. the conditional probability $p(m|n)$. In a Friedman-Robertson-Walker model, it was defined via the transition probability,

$$p(m|n) \equiv \left| {}_{\rm in}\langle 0_{-\vec{k}}, n_{\vec{k}} || m_{\vec{k}}, r_{-\vec{k}} \rangle_{\rm out} \right|^2 . \tag{9}$$

Here $\vec{k}$ is the momentum of the observed quanta. Observe that in this definition it was tacitly assumed that no quanta are emitted in a direction opposite to the observer. With the aid of the Bogolyubov transformation formalism [20], it was shown that the resulting conditional probability is a negative binomial distribution [22]:

$$p(m|n) = \binom{m}{n} (1-x)^{(n+1)} x^{(m-n)} , \tag{10}$$

where x is the noise intensity $x \equiv (|B|/|A|)^2 \leq 1$ (A and B are the Bogolyubov coefficients between 'in' and 'out' modes and satisfy $|A|^2 - |B|^2 = 1$). The above definition distribution yields the following relation between the mean number of emitted and detected quanta

$$\langle m \rangle = \langle n \rangle + \frac{x}{1-x} (\langle n \rangle + 1) . \tag{11}$$

This result teaches us that as the signal travels through space–time, owing to the spontaneous and stimulated emissions it induces, quanta are added to it in a stochastic manner, jamming its informational content.

With these results in hand it is possible to obtain the channel capacity formula [eq. (5)]. Unfortunately, we had to resort to an approximation since it was very hard to calculate sums involving the logarithm of a binomial: this logarithm as a function of n, was fitted by a parabola. The result represents a good estimation in the adiabatic limit where $x << 1$:

$$H_{\max} = \ln(\langle m \rangle + 1) + \ln\left(\frac{\langle m \rangle + 1}{\langle m \rangle}\right) - \nu - \mu \langle m \rangle . \tag{12}$$

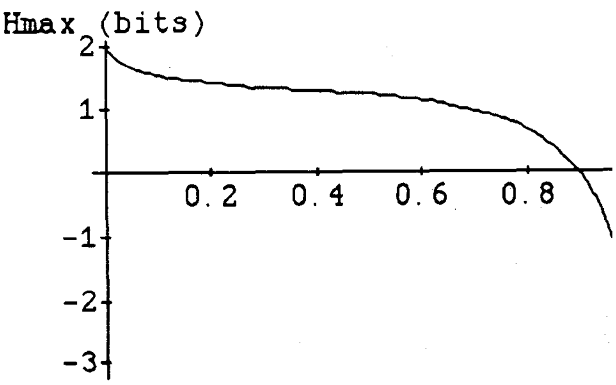
where

$$\nu = (4\ln 2)\, x^2 - x \ln x - (1-x) \ln(1-x) \tag{13}$$

and

$$\mu = (4 \ln 2)\, x + \nu . \tag{14}$$

This formula represents an upper bound on the amount of information that could, in principle, be transferred in a curved space–time. It is instructive to plot $H_{\max}$ against x for pulses containing, say, $\langle n \rangle = 1$ quanta.



Information borne by pulses containing originally one (mean) quantum as a function of the noise parameter

Observe that the amount of information becoming degraded by the noise increases very fast with x. Inspection of this figure shows an unphysical tail at $x \approx 0.8$, where $H_{\max} < 0$. This feature is a consequence of the adopted approximation. Indeed, we have argued [22] that $H_{\max}$ must vanish as $x \to 1$, because the quantity that measures the intensity of noise to which n emitted quanta are subjected, $N(n) \equiv -\sum_m p(m|n) \ln p(m|n)$, blows up in this limit. This means that *no information can be optically transferred* ($H_{\max} \to 0$) as $x \to 1$.

The last issue to be settled is how the noise parameter is related to space–time itself. We considered the information carriers to be scalar particles (pions) in a nearly homogeneous space-time (perturbations around a smooth space–time) and solved the Bogolyubov coefficients in an adiabatic approximation. The result is

$$|B(\omega)|^2 = \int d\eta d\eta' \tag{15}$$
$$\times \quad C^{\alpha\beta\gamma\delta}(\eta) C_{\alpha'\beta'\gamma'\delta'}(\eta') K^{\alpha'\beta'\gamma'\delta'}_{\alpha\beta\gamma\delta}(\omega, \eta, \eta');$$

here η is the conformal-time variable and K is the kernel:

$$K^{\alpha\beta\gamma\delta}_{\alpha'\beta'\gamma'\delta'}(\eta, \eta') = \frac{1}{(4\omega)^2} e^{i\omega(\eta - \eta')} \tag{16}$$
$$\times \quad a^2(\eta) a^2(\eta') n^\alpha(\eta) n^\gamma(\eta) k^\beta k^\delta n_{\alpha'}(\eta') n_{\gamma'}(\eta') k_{\beta'} k_{\delta'}$$

where n^α is the orthonormal vector to the space-like hypersurfaces and k is the normal vector pointing in the line-of-sight direction (η is the conformal time), $C^{\alpha}_{\beta\gamma\delta}$ is the Weyl tensor, which is a measure of the departure of space-time from homogeneity and isotropy (clumpiness) [23]. Of course, if the Weyl tensor grows, higher-order convolutions of the Weyl tensors with this kernel must be considered.

This equation tells us that if the signal travels through regions where the Weyl tensor is not negligible before reaching us, then a considerable fraction of information it bore is washed out, and the clumpier and the larger these regions are, the more information is degraded [1]. This situation is very akin to the information loss that occurs when we observe a landscape on a foggy day. For this reason we called this gravitational noise a quantum fog [25].

A quantum cosmic censorship?

Penrose's cosmic censorship remained an important puzzle in general relativity [24]. This conjecture was so important that, for many and many years, much attention and effort was diverted into geometric and local methods which could rule out the possibility that a naked singularity would be formed as the endpoint of a gravitational collapse. Unfortunately, this conjecture is now contradicted by many counter-examples [26]-[30]. Do we have to live with the fact that a space time singularity could be observed, and that predictability might be at stake?

[1] Eq. (16) was derived under the assumption that space-time is homogeneous. However, if the scale at which the Weyl tensor changes in space is much larger than the size of the pulse, eq. (16) should yield a good approximation for non-homogeneous space-time, provided the Weyl tensor is evaluated along the pulse world-line.

In the light of these counter-examples, we recently proposed a quantum mechanical version of the cosmic censorship hypothesis [25], which acknowledges that singularities that are not clothed by an event horizon might exist but cannot, however, be observed. A pulse originating from the vicinity of a space–time singularity has to travel through regions where the Weyl tensor is very large (actually diverges). Accordingly, close to the singularity a very intense (quantum) noise is produced ($x \to 1$) and all the information carried by the signal is washed out during its journey toward an asymptotic observer, i.e., $H_{\max} \to 0$. Thus, in contrast with black holes, where singularities are hidden by an event horizon, naked singularities are shielded by a cloud of a very intense quantum fog (noise) which precludes its observation. So it seems that we can neither learn anything about the singularity nor notice the breakdown of predictability. However, there are observables whose information is not conveyed by quanta and that are detected either via the Gauss theorem (electric charge, angular momentum and mass) or via a Bohm-Aharonov experiment (topological quantities). This fact raises the intriguing possibility that, after all, the 'no-hair' theorem [4,5], which was originally conceived for black holes, is more fundamental than thought earlier: it might well apply to naked singularities.

Unfortunately there are some pieces missing in the above resolution of the cosmic censorship problem. The first is that, as the singularity is approached, the adiabatic approximation is no longer appropriate and the Bogolyubov coefficients we obtained can no longer be relied upon. The situation is far more serious than it might appear, because it is a matter of principle: since we are unable to specify the boundary conditions at the singularity, we do not know how to calculate these coefficients between states at the singularity and some asymptotic region of space–time. If it could be shown that the adiabatic approximation represents a lower bound for these coefficients, regardless of the actual boundary condition (which is anyway unknown), then we would be in business again, because the noise actually produced would always larger than what we predicted and $H_{\max}$ represents an upper bound on the actual information conveyed by the signal. Another important issue that remains to be settled is the question of whether a different choice of representation than the 'number of particles' states could improve communication. Holevo's [31] theorem seems to support the view that the number-of-particles representation optimizes the channel capacity with respect to the class of states, i.e. no other class of states that could be used for signaling could do better than this. We would have preferred to reach similar conclusions by taking variations of $H_{\max}$ [eq. (5] with respect to the class of states.

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