Invariant Quantum Ensemble Metrics

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ABSTRACT

We recently discovered when two ensembles of N qubits are initialized to random phases, then the Cartesian distance metric between the ensemble states is approximately $\sqrt{2N}$, with a standard deviation of ½. This research relates inner product and quantum ensemble metrics to the "*standard distance*" metrics defined by *correlithm object theory*.

Correlithm object theory describes how randomly selected points (or COs) in high dimensional bounded spaces can be used as soft tokens to represent states at the ensemble level. The initial CO research was performed using unit N-cubes, but under Air Force SBIR contract, we extended this theory for other bounded metric spaces such as binary N-spaces, complex N-spaces and Hilbert spaces. A quantum encoded CO can be created by initializing an ensemble of qubits based on a phase ensemble of uniformly distributed, randomly chosen phases with values from 0 to 2π.

If a quantum encoded CO token is created as a qubit ensemble **Q** initialized using a specific phase ensemble **P**, upon quantum measurement using another phase ensemble of basis states **B** it produces a binary ensemble **A** of measurement answers. Multiple trials using the exact same ensembles **P** and **B** generate answer ensembles that are correlated with each other, since their distance metric is 70.7% of the binary standard distance of $\sqrt{N/2}$. This means that quantum encoded CO tokens survive this measurement process. In contrast, choosing new random ensembles **P** or **B** for each trial generates uncorrelated answer ensembles. This paper describes and demonstrates how quantum measurement acts as a noise injection process from the correlithm object perspective.

Keywords: quantum correlithm objects, standard distance, soft tokens, quantum measurement, noise injection, tokens survive measurement, probabilistic geometry

1. INTRODUCTION

The idea of relating distinguishable states and statistical distance metrics for quantum ensembles was discussed in the influential paper "Statistical distance and Hilbert space" [1]. Their definition of statistical distance is the count of the distinguishable states between two given states. Wootter's paper defined the problem of using a finite ensemble of identically prepared quantum systems to distinguish (in a fixed number of trials) between two slightly different preparations. This is problematic for a low number of trials because of the unavoidable statistical fluctuations due to quantum measurement. Wootter showed that the statistical distance is unexpectedly the same as the angle between the states. The angle is the only Riemannian metric that is invariant under all unitary transformations.

This idea was refined further in another paper "Information and Distinguishability of Ensembles of Identical Quantum States" [2]. This paper shows that by using Shannon's 12th theorem, the number of distinguishable states for a single qubit is $W(\alpha_1, \alpha_2, n) = |\alpha_1 - \alpha_2| \sqrt{\frac{2N}{\pi e}}$. The two angles are α_1 and α_2 while N is the number of identically prepared quantum states forming either ensemble. The authors' requirement for success was that the ability to distinguish between two states would approach 1 as the number N of identical state copies in the ensemble approaches infinity. The authors also defined the number of distinguishable states of larger sized Hilbert spaces formed by multiple qubits. The reason behind this exercise is to encode the relevant state as a qubit phase, which is not observable except thru sampling, due to the statistically nature of quantum measurement.

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Both of the previous papers make the supposition that each of the N qubits defining the ensembles are identically prepared to the same phases either α_1 or α_2. The major idea discussed in this paper is to initialize each of the qubits in an ensemble to *randomly chosen phases* rather than the same phase. This simple variation encodes the state at the ensemble level rather than the individual qubit phase, which allows many more unique states and produces other interesting properties. An ensemble of N qubits can then be viewed as an N-dimensional space, since each qubit acts as an independent cell or dimension with a phase value in the range of 0 to 360 degrees. This kind of N-dimensional space is different than the tensor product space of the entangled quantum register. We call a randomly generated point defined in any N-dimensional ensemble a *correlithm object* (or CO). For COs in random ensembles, Correlithm Object theory describes metrics and properties that are invariant of the range/type of cell values.

This paper introduces some of the metrics and information theoretic properties of COs encoded as random phase quantum ensembles. Most notably, a qubit ensemble **Q** initialized using a specific phase ensemble **P** produces a binary ensemble **A** of measurement answers, upon quantum measurement using another phase ensemble of basis states **B**. Multiple trials using exactly the same ensembles **P** and **B** generate answer ensembles that are correlated with each other since their distance metric is 70.7% of the expected binary standard distance of $\sqrt{N/2}$. This means that quantum encoded CO tokens survive this measurement process, since this process is phase invariant. In contrast, choosing new random ensembles **P** or **B** for each trial generates uncorrelated answers. This paper describes and demonstrates how quantum measurement acts as a noise injection process from the correlithm object perspective.

2. RANDOM VALUED ENSEMBLES

The book "Correlithm Object Technology" [3] describes Correlithm Object theory extensively. It defines COs as randomly chosen points in bounded N-spaces, where usually N > (~20). An ensemble of N cells, dimensions, indexes or addresses, can be viewed as a bounded N-space, where each dimension contains a bounded value. Three different kinds of N-spaces must be defined to discuss quantum ensembles: phase ensembles with real number cell values between [0°..360°], binary N-spaces with binary valued cells [0, 1] and qubit ensembles with each qubit initialized to a random phase. This paper starts with the process for the generation of the prototypical CO concepts and metrics using unit N-cubes with real number cell values between [0..1] and then varies that process for these other kinds of cell values.

To build an ensemble, first establish the ensemble size N, for example N=100, then randomly pick uniformly distributed values (or other distributions) for each of the N cells and label that point in the space 'A'. Repeat the random generation process for another point 'B'. Compute and record the Cartesian distance between points A and B (see Figure 1).

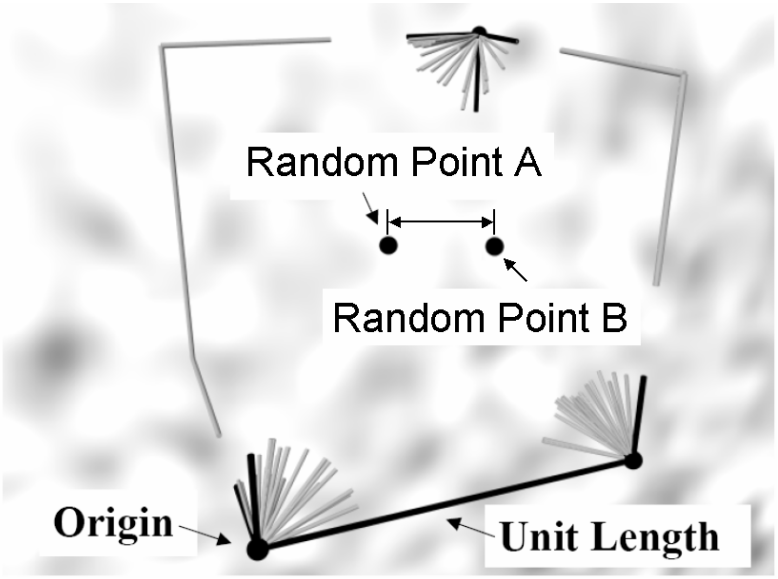

Figure 1 Unit N-cube containing two random points A and B

Performing this process with cells of real values [0..1] for I=10^4 iterations creates a distribution with the average expected distance of $\approx \sqrt{N/6} \Rightarrow 4.082$ and standard deviation of $0.2415 = \sqrt{7/120}$. We call this expected distance metric *standard distance*. Repeating this process for N=200 produces the average distance of $\approx \sqrt{N/6} \Rightarrow 5.77$ and the same standard deviation. Figure 2 contains the combined histogram plots for many values of N.

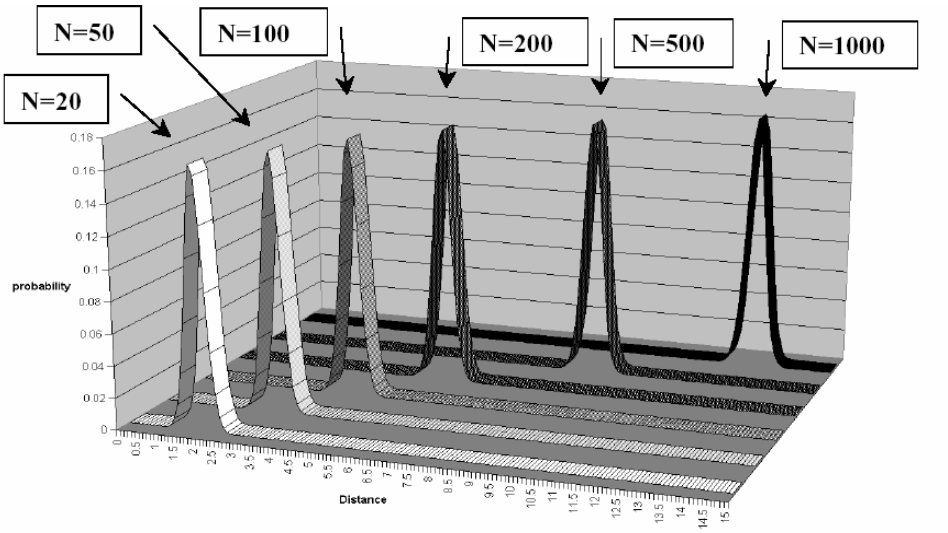

Figure 2 Histograms of unit N-cube Standard Distances for N= 20, 50, 100, 200, 500 and 1000

This is an unintuitive result. Any two randomly chosen points in a unit N-space are approximately the same distance apart (for a specific N). That distance is approximately $\sqrt{N/6}$. This raises an important question. Given a specific sized ensemble N, how does a virtually constant Cartesian distance metric emerge from purely random values? A quick peak at the mathematics provides the understanding. Represent two points as **A** = [a_1, a_2, …, a_N] and **B** = [b_1, b_2, …, b_N]. The Cartesian distance is cdist(**A**, **B**) $= \sqrt{\sum (a_i - b_i)^2} = \sqrt{\sum (x_i)^2}$. Since $x_i = a_i - b_i$ is the difference of two random numbers it is also a random number, and its square $y_i = x_i^2$ is another random number, both with different bounds and means. Table 1 details how these bounds, distributions and averages change as the computation proceeds. The last four columns were produced by building two representative points **A** and **B** using N = 10^6. These statistical results match the analytical results produced using symbolic polynomial expansions in Mathematica (see Appendix I in [3]).

Table 1 Bounds and Distributions for steps in distance computation

Step	Expression	Distribution for real values 0→1			
		lower	upper	average	distribution
1	a_i, b_i	0	1	≈ 1/2	uniform
2	$x_i = a_i - b_i$	–1	1	≈ 0.0	not uniform
3	$\lvert x_i \rvert$	0	1	≈ 1/3	not uniform
4	$y_i = x_i^2$	0	1	≈ 1/6	not uniform
5	$z = \sum y_i$	≈ N/6			single value with standard deviation
6	$d = \sqrt{z}$	Cartesian distance(A, B) $\approx \sqrt{N/6}$			
7	$e = z/N$	Expected value or Avg of $y_i \approx 1/6$			

How the average of the distributions change as the computation proceeds from the original random values is the key to understanding how an expected distance results from a specific N. Step 5 is the key because the sum *z* is used to compute the both the Cartesian distance *d* or the average *e*. Therefore, in order to compute a Cartesian distance for random points, an intermediate sum causes the cell level randomness to cancel out, just as when computing the average. The last step is the only difference between computing an average (step 7) or a Cartesian distance (step 6). A similar intermediate sum occurs when computing the inner product computation again leading to nearly constant ensemble phase angle results using randomly selected cell values.

Now repeat this process using binary cell values (of 0 *or* 1) for, say, $I = 10^6$ iterations. This average distance between random binary ensembles is $\approx \sqrt{N/2} \Rightarrow 7.07$ with a standard deviation of $0.333 = \sqrt{1/9}$. Repeating this for N=200 produces the average distance of $\approx \sqrt{N/2} \Rightarrow 10.0$ and the same standard deviation. Since the cell values are binary, the distances are not continuous, but result in only the square roots of integer values. The binary values are effectively randomly chosen corners of the unit N-cube and the distances follow the binomial distribution of Pascal's triangle.

The key to understanding these results is to explore the geometry and topology of randomly chosen points inside the unit N-space, as well as its random chosen corners. Since all the points/corners are at approximately equal distance to all other points/corners, the essential topology here is a high dimensional version of a tetrahedron (which is quite hard to visualize for N>3). Perhaps the best way to visualize this geometry is as a roughly spherical cloud of points, where the points are all about equally spaced. The key relevant landmarks are any two opposite corners of the N-space (such as C = [0, 0, …, 0] and O = [1, 1, …, 1]) plus the midpoint of the space (such as M = [½, ½, …, ½]). The main diagonal has a length of cdist(C, O) = $\sqrt{N}$ and the half diagonal is cdist(C, M) = cdist(M, O) = cdist(C, O)/2 = $\sqrt{N}/2 = \sqrt{N/4}$.

Now compute the Cartesian distance from each randomly chosen point inside the unit N-space to the midpoint and the expected distance for N=100 is $\approx \sqrt{N/12} \Rightarrow 2.886$. Compute the distance from any randomly chosen corner to the midpoint, and the expected distance is $\approx \sqrt{N/4} \Rightarrow 5$. These metrics have constant standard deviations. Both of these distances can be thought of as a radius for the respective high dimensional tetrahedron, which we call the *standard radius*. Figure 3 depicts how all randomly chosen points (or COs) are approximately the same distance from the midpoint M of the unit N-cube space. Also depicted are the randomly chosen corners of the unit N-cube space (or binary space). Remember, even though this figure cannot illustrate it, all related COs shown are approximately equally spaced forming two high-dimensional tetrahedrons of different sized radius. In addition, all randomly chosen COs are approximately the same distance from corners C and O ($\approx \sqrt{N/3}$ for random points and $\approx \sqrt{N/2}$ for random corners).

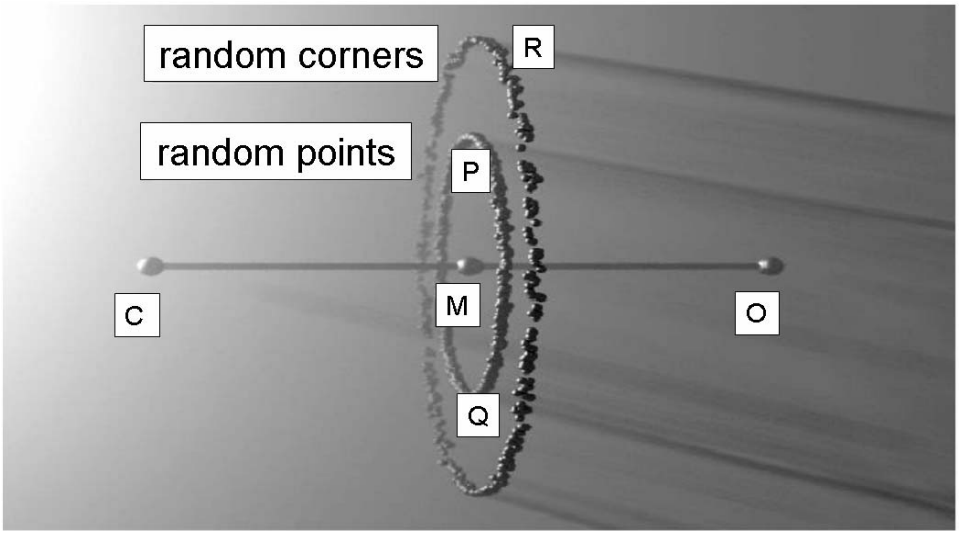

Figure 3 Distances of random corners and points from perspective of corners C & O and midpoint M

The columns labeled *unit edge* in Table 2 summarize the various metrics we have just described. The rows are organized with larger metrics listed first, where the shortest is the standard radius. Dividing by the standard radius normalizes all these metrics and produces the two columns labeled *unit radius*. This ensemble metric normalization procedure (unit radius) is mathematically similar to the unitarity constraint imposed on qubit Hilbert spaces.

Table 2 Unit Edge and Unit Radius Standard Metrics

Type	Unit Edge		Unit Radius (Normalized)	
	Cartesian Distance	**Standard Deviation**	**Cartesian Distance**	**Standard Deviation**
Major Diagonal: maximum corner to corner = cdist(C, O)				
Exact	$\sqrt{N}$	0	$\approx \sqrt{12}$	≈ 0
Corner Distance: random corner to random corner = cdist(C, R) = cdist(O, R)				
Probabilistic	$\approx \sqrt{N/2}$	$\approx \sqrt{1/9}$	$\approx \sqrt{6}$	$\approx \sqrt{4/3N}$
Corner-Point Distance: random corner to random point = cdist(C, P) = cdist(C, Q)				
Probabilistic	$\approx \sqrt{N/3}$	$\approx \sqrt{1/15}$	$\approx \sqrt{4}$	$\approx \sqrt{4/5N}$
Half Diagonal: midpoint to corner = cdist(M, C) = cdist(M, O) = cdist(C, O)/2				
Exact	$\sqrt{N/4}$	0	$\approx \sqrt{3}$	≈ 0
Standard Distance: random point to random point = cdist(P, Q)				
Probabilistic	$\approx \sqrt{N/6}$	$\approx \sqrt{7/120}$	$\approx \sqrt{2}$	$\approx \sqrt{7/10N}$
Standard Radius: midpoint to random point = cdist(M, P) = cdist(M, Q)				
Probabilistic	$\approx \sqrt{N/12}$	$\approx \sqrt{1/60}$	$\approx \sqrt{1}$	$\approx \sqrt{1/5N}$
Edge: length of size				
Exact	1	0	$\approx \sqrt{12/N}$	≈ 0

We find this result amazing, because all normalized standard metrics are constants, the square roots of small integers or ratios. Remember, these results are *approximately constants*, because they actually are statistically derived and have a non-zero standard deviation. Notice how the normalized standard deviations all contain N in the *denominator*! This means the normalized standard deviations approach 0 as the size N of the ensemble increases. This table can be illustrated as the graphic in Figure 4, where 'C' represents a random *c*orner, 'O' the *o*pposite corner, 'M' the *m*idpoint and 'R' a *r*andomly chosen corner. Random CO points 'P' and 'Q' are shown with unit radius distances to midpoint M.

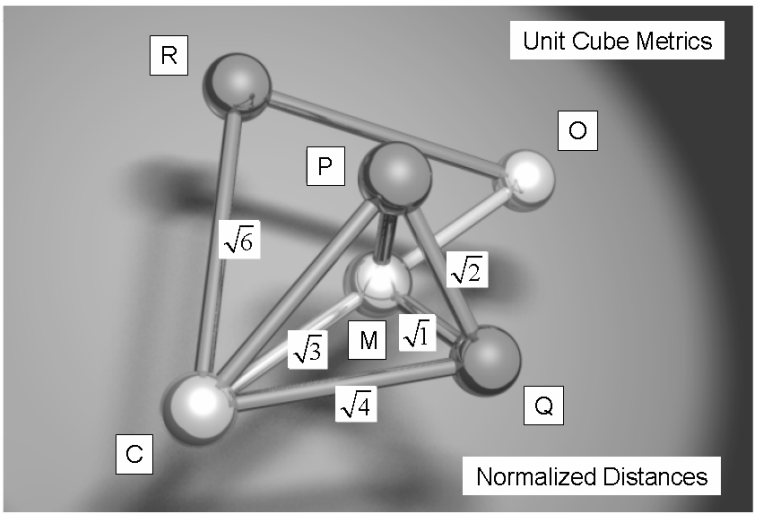

Figure 4 Geometrical representation of Unit N-cube normalized distance metrics

This graphic makes one fact more apparent. Random points are approximately orthogonal! Consider triangle PQM. The length of line PQ, the normalized standard distance metric, is $\approx\sqrt{2}$. It forms the hypotenuse of the triangle with the other two sides the unit radius $\approx\sqrt{1}$. These normalized metrics form a right triangle, where the two points P and Q can be considered the end points of vectors. If the starting end of the vectors is the midpoint M then these standard radius vectors are almost orthogonal. This probabilistic angle metric has less variation with increasing N due to decreasing standard deviations. Figure 5 illustrates the angle observed to be ~90° (or ~41.4°) depending on the reference point of the vectors as either the midpoint (or corner) of the unit N-cube. Many other angles are also nearly orthogonal. Because of the geometry we can show that the angle between two random COs is proportional to the distance between them.

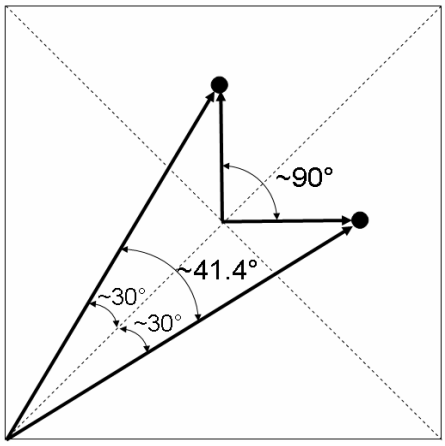

Figure 5 Probabilistic angle for two COs from midpoint and corner perspectives

From the perspective of the center of the high-dimensional tetrahedron (usually the midpoint of the space) vectors formed using *any two* random COs are nearly orthogonal! Symmetric spaces (where the midpoint M = [0, 0, …, 0] automatically makes this the default perspective. This property of near mutual orthogonality of randomly generated points means that ensemble spaces can produce a very large number of CO points that become closer and closer to orthogonal with increasing N. For example, with N > 3000 the standard deviation of the 90° angle between *any two randomly selected COs* is < 1°.

These normalized CO metrics are invariant to the scaling size or displacement of the bounded spaces and have been observed for all bounded spaces we have investigated during this research. These CO metrics define a measure for how random something can be compared to maximum randomness, which is known as the *standard distance* of CO theory.

3. QUANTUM ENSEMBLES

This section describes the phase ensemble and qubit ensemble. A phase ensemble is simply an array of randomly generated phases that are stored classically so the values can be used repeatedly. The phase ensemble is identical to the unit N-cube where the values in the range [0..1] are mapped to phase angles [0°..360°] (or [0..2π]). The only difference is these values represent a phase ring that circles back to the starting location where 0°=360°. This ring means two distances are possible, the short way and the long way around the circle. If the shortest distance were always chosen, then the maximum phase difference between two randomly chosen phases would be ½ of the overall range or 180°.

We can determine the impact of using the shortest distance. For example, using an ensemble size of $N=10^4$, the average phase difference (using shortest route) between two random phases in the range of [0°..360°] is ~¼ of that range or ~90°, which is slightly smaller than the expected difference of ~1/3 (see step 3 in Table 1) the range of the non-wrapped distances. If the four quadrants of phase are mapped to quadrant 1 by ignoring the coefficient signs (allow only [0°..90°]), then the expected angle difference is ~22.5° = 90°/4. Just from this analysis, we could conclude that large random phase ensembles of qubits may be useful since the average phase difference is not 45°. A phase difference of 45° would produce the maximum amount of randomness for quantum measurement.

Now we define a quantum ensemble **Q** as an array of N qubits **Q** = [q_1, q_2, …, q_N], where each q_n is initialized using a phase gate using a randomly generated phase. Create two quantum ensembles A = [a_1, a_2, …, a_N] and **B** = [b_1, b_2, …, b_N] and compute the standard radius and standard distance. Since the internal complex valued states of the qubits are not directly accessible in physical quantum systems, the distance metrics on quantum ensembles can only be computed using tools that model qubit ensembles. We have built quantum modeling libraries in Python, C++ and Block Diagram Toolset[1] during the SBIR contracts. The Block Diagram tool will be demonstrated later in this presentation.

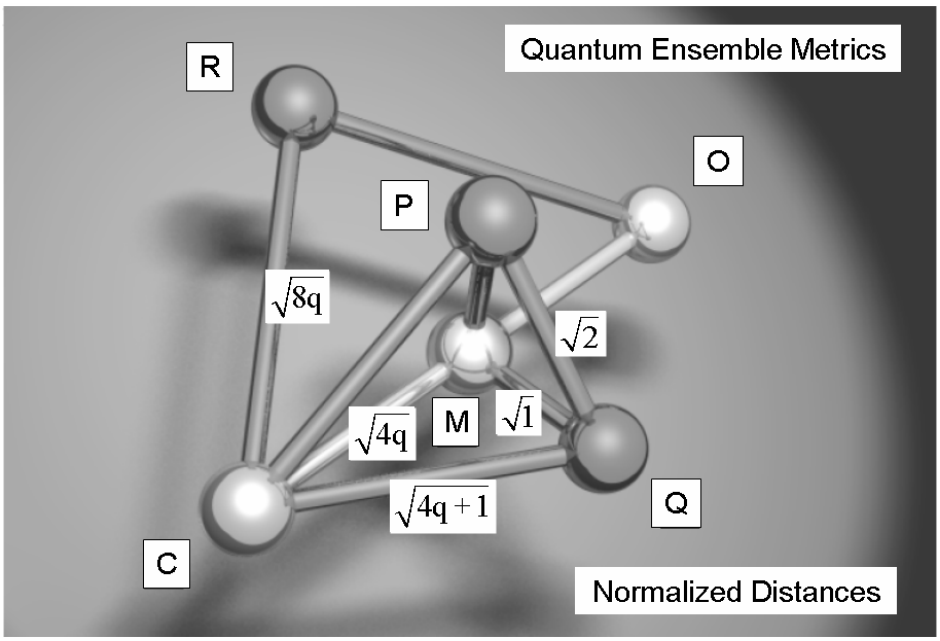

Figure 6 Geometrical representation of Quantum Ensemble normalized distance metrics

The bounding box for a qubit ensemble is a symmetric space bounded by [–1..+1]. Each qubit is defined by two complex numbers meeting the unitarity constraint. Cartesian distance can be computed for complex numbers of qubit states as well as real numbers. Due to unitarity constraint, where the radius for each qubit is 1, the expected standard radius for a quantum ensemble of N qubits is simply $\sqrt{N}$, which is the same size as the major diagonal of the unit N-cube. CO theory predicts, that the standard distance is $\sqrt{2}$ larger than the standard radius, or $\sqrt{2} * \sqrt{N} = \sqrt{2N}$. This is confirmed with measurements on modeled qubit ensembles. The standard deviation of the standard distance is measured to be ½. Interestingly, if each cell in the ensemble is a quantum register of q>1 qubits, then the standard distance and standard radius remain the same, because a quantum register of any size has the unitarity constraint applied, essentially pre-normalizing each cell radius to 1. In general, the standard deviation for any quantum ensemble is $\sqrt{1/2}^{q+1}$, where q is the (same) number of qubits in each cells' quantum register. Figure 6 summaries the normalized metrics for quantum ensembles where q is the number of qubits in each of the cells.

4. COS SURVIVE MEASUREMENT

Now create a quantum ensemble **Q** = [q_1, q_2, …, q_N] as an array of N qubits, where each qubit q_n is initialized using a phase gate to the corresponding phase p_n from the phase ensemble **P** = [p_1, p_2, …, p_N]. When each q_n is measured from the corresponding basis angle b_n, where **B** = [b_1, b_2, …, b_N] is another random phase ensemble, then the corresponding binary values a_n are recorded in the binary answer ensemble **A** = [a_1, a_2, …, a_N]. Figure 7 illustrates these ensembles. Incidentally, since the randomness of **P** makes this process phase invariant (all phases are present), then the cells of **B** can all have the same basis phase value without changing the results. Randomizing **B** does not increase the randomness.

[1] Block Diagram Toolset is a product of Hyperception of Dallas, Texas.

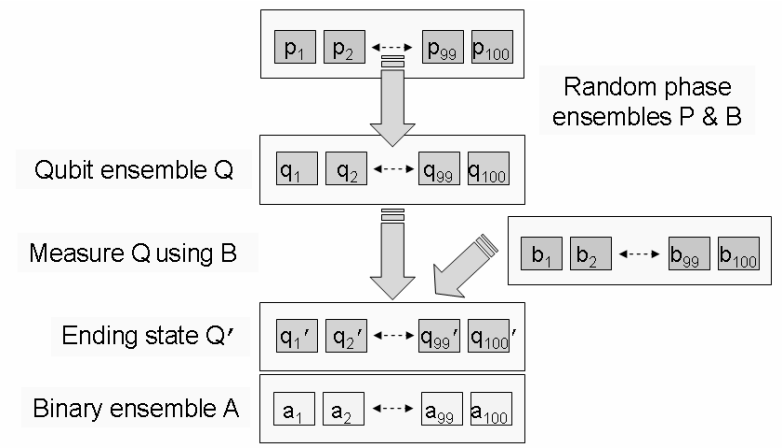

Figure 7 Pictorial view of ensembles P, B, Q, Q' and A

We just have shown it is possible to encode classical COs as quantum COs by converting random values to phases. It is also possible to decode a quantum CO back into a classical CO or binary bit array by *measuring* the quantum CO over multiple trials. If the above process is repeated I times while cell-wise averaging the recovered binary ensembles, then the resulting real unit N-cube ensemble **U** = [u_1, u_2, …, u_N] contains the probabilities $u_n = \sum_i a_{ni} \Big/ I$ related to the measurement of q_n using basis b_n. This average of many trials produces the average of 1s/0s that represents the q_n and b_n phase difference as a probability. Assuming **U** is known, the distance measure between any **A** and **U** is approximately the standard radius for that cluster. It can be determined with high likelihood in a *single trial* (with N>(~20)) if a quantum encoded CO **P** was transmitted thru this process if either **A** or **U** are previously known and we assume that there are no other noise sources. We believe this result is so significant that we have filed a provisional and final US patent on the idea. Here is the sequence of steps and the process details.

First, randomly choose a phase ensemble **P** with N=100 and store away for reuse later. We choose N=100 for illustration but any N>(~20) can be used. Second, initialize the qubit ensemble **Q** by applying the phases in **P** to phase gates. Third, choose a random (or constant) phase ensemble **B** with N=100 and store away for reuse later. Measure ensemble **Q** using the basis angles from ensemble **B** and store the binary answers in binary ensemble **A**. As expected, **A** appears to have a random number of 1s and 0s because generally random-in gives random-out. Remember these results are due to the averaged results of a uniform phase qubit ensemble as measured from another uniform phase basis ensemble. The next step is to use exactly the same classically stored ensembles **P** and **B** for another trial. Figure 8 graphically shows the setup for two trials.

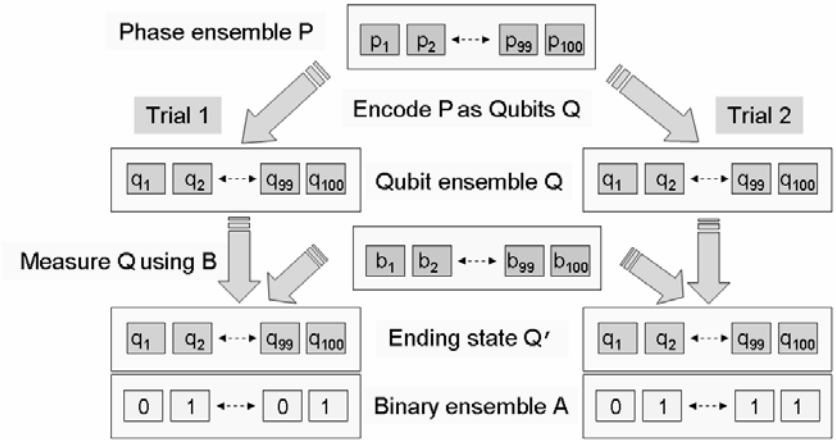

Figure 8 Experimental setup for Quantum COs survive measurement

Compute the distance between the answers of the two trials, which will be a distance approximately 70.7% of the expected standard distance of $\sqrt{N/2}$ for binary ensembles. Repeat these ensemble pairs for I=1000 iterations and the normalized distances produces the distributions displayed in the Figure 9. This experiment was run for N=100 and N=1000. Notice how the normalized discrete binary distances show up as explicit bands on the histogram of N=100.

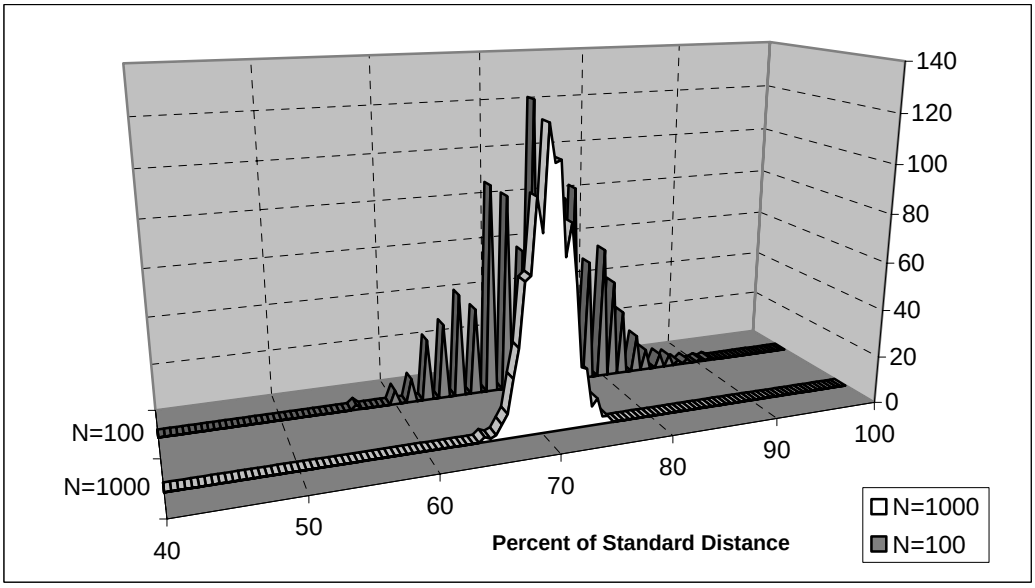

Figure 9 Histogram of Normalized Distances showing COs survive measurement

Figure 10 shows the same results produced by running this experiment in the commercial tool called Block Diagram using our custom quantum libraries with N=200. This screen capture illustrates that distance between random tokens (as the right most Gaussian on lower histogram display) are at standard distance $\sqrt{N/2}$ and correlated tokens are 0.707 of standard distance (as the upper Gaussian display and left most Gaussian on lower display).

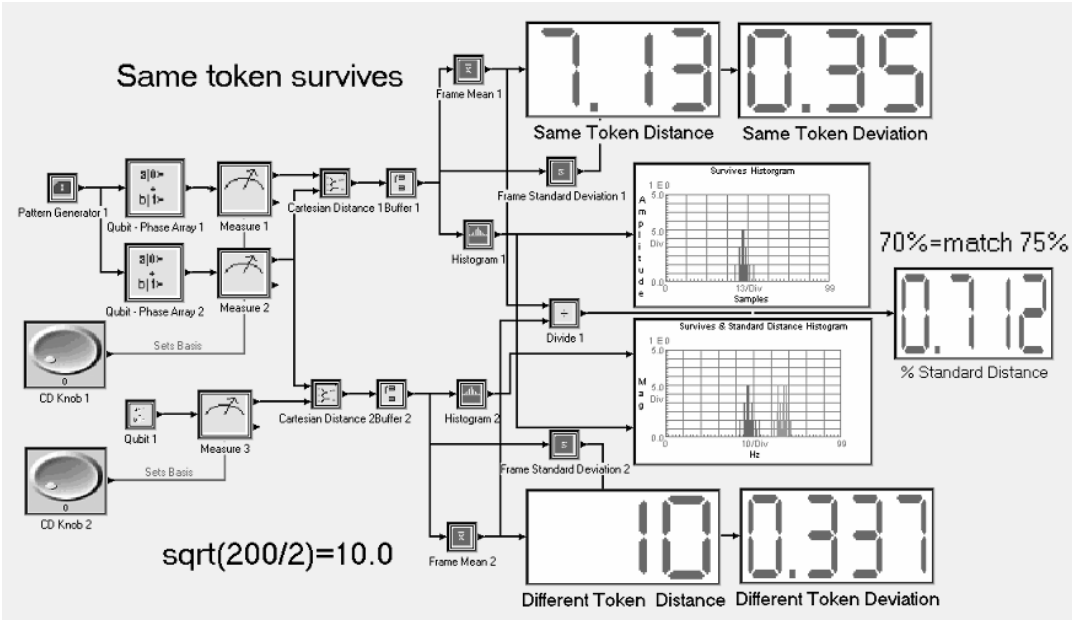

Figure 10 COs survive Measurement using Block Diagram Tool

This result means that multiple trials produce *correlated* binary ensembles! The amount of 70.7% of standard distance means that 75% of the cells have the same value (compared to 50% for uncorrelated ensembles). This is reasonable since ~50% of the uniform phase qubits are producing noise-like patterns (around the 45° phase difference) and ~50% are producing constant-like results (away from 45° phase difference). A single pair of ensembles $\mathbf{P_0}$ and $\mathbf{B_0}$ produces clustered results **X** from multiple trials. Likewise another pair of ensembles $\mathbf{P_1}$ and $\mathbf{B_1}$ produces clustered results **Y** from multiple trials that are standard distance away from cluster **X**, as shown in Figure 11. Because of these distances, answers from the X cluster are easily distinguished from answers from the Y cluster when N is large enough.

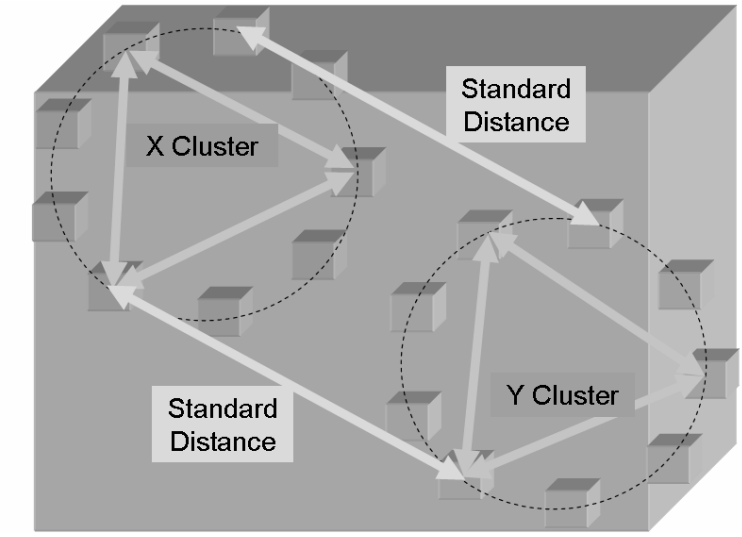

Figure 11 Cluster distances proving that COs survive measurement

The midpoint of each cluster is the expected value for related trials and is a unit N-cube ensemble **U** containing the expected probabilities. The measurement process introduces noise that generates points with a standard radius from that expected midpoint **U**. The points in each cluster are only 70.7% of the global binary standard distance apart from each other. A CO point in one cluster is standard distance apart from any CO point in another cluster. This localized clustering of points due to measurement indicates that this process introduces less than the maximum noise into the uniform phase qubit ensembles, so the results are correlated. Therefore, a quantum encoded CO token subjected to this measurement process is easily distinguished from other standard distance CO tokens and therefore *survives* measurement. In contrast, choosing new ensembles **P** or **B** for each trial generates uncorrelated answers and states. Quantum measurement acts as a noise injection process into a quantum CO's phase invariant representation.

5. APPLICATIONS

Our company is creating general-purpose computational systems using correlithm object technology. The word "Correlithm" is a contraction made of the words "*Correl*ational Algor*ithm*". Correlithm Object theory was developed as a model for how brains represent and manipulate information using natural randomness of neuronal connectivity and firing. Bundles of neurons are readily viewed as very large ensembles. Likewise, groups of photons produced by laser (or molecules in NMR) are readily viewed as very large ensembles. Each ensemble can be thought of as a separate space and operations can transform COs in one subspace into other CO tokens in another space.

For example, Figure 12 illustrates how the "NOT" operation is represented using two spaces s1 and s2, each with predefined random CO tokens labeled T and F. Our CO "rules" based only on distance metrics can map any state near either of the CO tokens in s1 to the equivalent CO tokens in s2. The rules for this "NOT" operation can be expressed as the illustrated pair of linearly independent and concurrent state transformations. We have also shown that sets of CO rules are possible for other logic operations (including AND, OR, XOR & LATCH) but not restricted to purely classical computation states. Thus CO theory is both Boolean and Turing Complete.

s1.T >: s2.F
s1.F >: s2.T

NOT rules

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

NOT operator

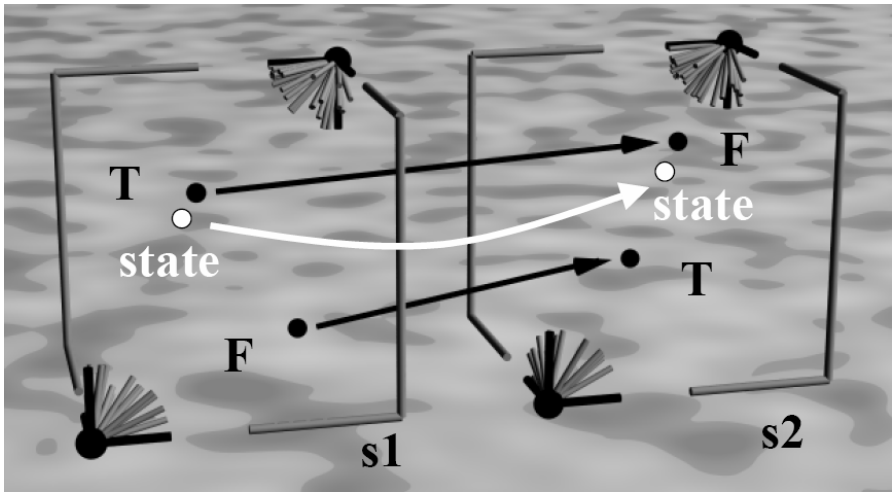

Figure 12 NOT operator state mappings using subspaces, CO tokens and rules

Since the CO tokens are nearly orthogonal, it is possible to express a state that is the combination of tokens T and F as a point approximately half the distance between them and at standard radius from the space midpoint. This mixed token is the classical equivalent of quantum superposition but is better known in the literature as code division multiplexing. Therefore, as with qubits, CO systems can naturally represent the *superposition* of states. This programming paradigm is a fertile area in which to explore state transition systems that contain both classical and quantum concepts. In fact, the NOT gate example from above is actually equivalent to the matrix version of the quantum NOT operator and works correctly for both classical states (mutual exclusion of states T/F) or quantum states (superposition of states T/F). Remember, this CO rule programming style works for any ensemble with either classical or quantum cells.

This result is significant because large random (yet repeatable) qubit ensembles should be much easier to build (using for example a laser and various thicknesses of optical glass) compared to large entangled quantum registers. Since quantum ensembles can evidently represent classical states and perform classical operations, this CO approach may be useful as a fault tolerant solution for classical computers as semiconductor process scaling continues into the 10s of nanometer size. CO theory insures that it is highly unlikely that two tokens could be significantly closer than standard distance. So if a state does appear near a CO token it is almost certainly produced by the same process that generated the original token, even if some amount of noise has been inserted. For this reason CO tokens are considered to be soft or fuzzy tokens that allow robustness in the face of noise (due to manufacturing or operation). They can also represent mixtures of states in superposition, which at times may be considered to be paradoxical. Both of these properties are found in human cognition. Quantum encoded CO ensembles may be an interesting and pragmatic alternative to qubit encoded states that rely on reversible logic gates for classical operations and ebit entanglement for error correction.

Even though CO systems can represent superposition we have not observed any CO property equivalent to the inseparable entanglement properties of ebits (or EPR bits). We have explored using ensembles of ebits using the amount of partial entanglement as the primary random variable for the ensemble, but we have no interesting results to report at this time.

6. CONCLUSIONS

We believe that ensembles of states are ubiquitous in nature in classical, neural, quantum and many other forms. By understanding that randomly chosen points in ensembles with N>(~20) always exhibit the standard distance and standard radius metrics, we obtain new analytical tools for evaluating randomness and correlations. Quantum theory and Correlithm Object theory are synergistic because each adds insight and richness to the other domain, even though they both depend on randomness, probabilities, bit capacity and distance metrics. Quantum superposition and matrix operators are naturally built into CO theory using tokens and rules but are not limited solely to unitary operations.

In fact, Correlithm Object Theory states that it is possible to use randomness as a *resource* for constructing data tokens. We have built a set of multiplatform tools and an application-programming interface to aid in the construction, modeling and execution of large CO systems and applications. We can build large sets of named tokens at standard distance (or less) and the corresponding rules to model systems that have fault tolerant or fuzzy logic behaviors using both classical and quantum cell types. Our original goal for developing CO theory was to show that CO systems act more like humans because they respond similar to examples they have previously been exposed to. We find it fascinating the same CO metrics and properties relate to ensembles in both neural and quantum systems.

We have shown that applying CO thinking and understanding to quantum ensembles leads to previously unrecognized properties of phase invariance in random phase quantum ensembles. We have shown that the standard distance for any quantum ensemble is $\sqrt{2N}$ and standard deviation of $\sqrt{1/2}^{q+1}$. We have also shown that random phase COs encoded in quantum ensembles survive measurement even when using random basis angles. This result leads to the quite interesting CO interpretation of quantum measurement as noise injection process. Because of these properties, it is possible to distinguish (with certainty approaching 1 as N increases) a specific quantum encoded ensemble in a *single trial* when measuring the qubits using a random basis angle(s).

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